Weighted average cost of capital (WACC) with risky debt: a simple exposition (I)[1]

Costo promedio ponderado de capital (CPPC) y deuda con riesgo: una presentación sencilla.

Custo médio ponderado de capital (CMPC) e dívida com risco: uma apresentaçao simples

Joseph Tham. [2]

Abstract

Debt is rarely risk-free. Yet, on grounds of simplicity, in most discussions on the weighted average cost of capital (WACC), we assume that the debt is risk-free. At the same time, in the calculation of the WACC, we may use a value for the cost of debt d that is higher than the risk-free rate r_f.

In this teaching note, using simple binomial models, we examine the weighted average cost of capital (WACC) with risky debt and no taxes. Taxes raise additional complications. In a subsequent note, we analyze the case with taxes. With risky debt, we have to use the expected rate of return on the debt rather than the promised rate of return on the debt in the formula for the WACC. Furthermore, we model the expected cost of risky debt as an increasing function of the amount of debt.

JEL codes D61: Cost-Benefit Analysis G31: Capital Budgeting H43: Project evaluation

Key words or phrases: Multiperiod WACC, Cost of capital, Risky debt.

Resumen

El costo de la deuda rara vez se puede considerar libre de riesgo. Sin embargo, en aras de la simplicidad, en la mayoría de las discusiones sobre el costo promedio de capital (CPPC) se supone que la deuda es libre de riesgo. Como resultado, cuando se calcula el CPPC se utiliza un costo de la deuda, d, mayor que la tasa libre de riesgo, r_f.

En este trabajo, usando un modelo binomial sencillo, se examina el costo promedio ponderado de capital (CPPC) con deuda bajo y sin impuestos. Los impuestos plantean complicaciones adicionales, lo cual supera el propósito del trabajo. Con deuda bajo riesgo se debe usar la tasa esperada de rentabilidad de la deuda, en lugar de la tasa pactada o contractual para incluirla en el CPPC. Más aun, se modela el costo esperado de la deuda como una función creciente del monto de la deuda.

Resumo

O custo da dívida, raramente, pode ser considerado livre de risco. Porém, por uma questão de simplicidade, na maioria das discussões sobre o custo médio de capital (CMPC) supõe-se que a dívida está livre de risco. Como resultado, quando se calcula o CMPC, utiliza-se um custo da dívida, d, maior que a taxa livre de risco, r_f.

Neste trabalho, usando um modelo binomial simples, examina-se o custo médio ponderado de capital (CMPC) com dívida baixa e sem impostos. Os impostos apresentam complicações adicionais, o que supera o propósito do trabalho. Com dívida sob risco, deve-se usar a taxa esperada de rentabilidade da dívida, em vez da taxa concordada ou contratual para incluí-la no CMPC. Mais ainda, modela-se o custo esperado da dívida como uma função crescente do valor da dívida.

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1. Artículo de reflexión sobre costo de capital.
2. Profesor visitante en la Escuela Sanford de Política Pública de la Universidad de Duke y el Centro Duke para el Desarrollo Internacional (DCID). Imparte cursos y talleres sobre evaluación de proyectos y análisis empírico para el desarrollo económico en la Universidad de Duke y en países en desarrollo en todo el mundo. Ver Social Science Research Network (SSRN). Duke Center for International Development DCID, Sanford School of Public Policy, Duke University ThamJx@duke.edu

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Palabras clave: CPPC periódico variable, costo de capital, deuda bajo riesgo

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Palavras-chave: CMPC periódico variável, custo de capital, dívida sob risco.

Introduction

Debt is rarely risk-free. Yet, on grounds of simplicity, in most discussions on the weighted average cost of capital (WACC), we assume that the debt is risk-free. At the same time, in the calculation of the WACC, we may use a value for the cost of debt d that is higher than the risk-free rate rf. There are several advantages in assuming that the debt is risk-free. First, when we assume that the debt is risk-free, we do not have to model the risk of default on the debt. Second, with risk-free debt, the expected return on the debt is equal to the promised return and we do not have to distinguish between the expected return on the debt and the promised return. Third, since the debt is risk-free, the cost of debt is constant and not a function of the amount of debt. Thus, we do not have to model how the cost of debt d varies as a function of the amount of debt D.

In this teaching note, using simple binomial models, we examine the weighted average cost of capital (WACC) with risky debt and no taxes. Taxes raise additional complications. In a subsequent note, we analyze the case with taxes. With risky debt, we have to use the expected rate of return on the debt rather than the promised rate of return on the debt in the formula for the WACC. Furthermore, we model the expected cost of debt risky as an increasing function of the amount of debt.

To reach a wider audience, the exposition is informal and technicalities have been kept to a minimum. We assume that the reader is familiar with risk-neutral probabilities. The paper is organized as follows. In Section One, we present the single period binomial model, in which we can solve for the cost of debt as a function of the amount of debt and obtain simple interesting relationships between the cost of debt, the return to levered equity and the amount of debt. In the single period binomial model with risk-free debt, the return to equity e is a function of the debt-equity ratio. With risky debt, in the single period case, the return to equity e is constant, and the cost of debt is a function of the amount of D. See Tham & Wonder (2002) for further details.

In Section Two, we present a multi-period model. In the multi-period model, the simple relationships that were obtained in the single period do not hold. Instead, we use one-way tables to show the relationship between the cost of debt, the return to levered equity and the amount of debt.

Section One: single period binomial model

We begin by reviewing the standard formula for the WACC. In words, the WACC is a weighted average of the market cost of debt and the return to equity, where the weights are the market values of the debt and equity, as percentages of the total value.

$$\text{WACC} = \%D*d + \%E*e \tag{1}$$

where d is the market cost of debt, e is the return to equity, %D is the market value of debt as a percent of the total value and %E is the market value of equity as a percent of the total value.[4]

Single period binomial model

We briefly review the single period binomial model. Consider a simple one-period binomial model with two states of nature. Let FCF(i,j) be the value of the FCF in the jth state of nature of the ith year. In the up state of nature, the value of the FCF is $190 and in the down state of nature, the value of the FCF is 50.

$$\text{FCF}(1,1) = 190 \tag{2.1}$$

$$\text{FCF}(1,2) = 50 \tag{2.2}$$

Let P be the set of probabilities for the two states of nature, where p^U is the probability of the up state of nature and $(1 - p^U)$ is the probability of the down state of nature. We assume that both states of nature are equally likely.

4. We have not put time subscripts for the parameters in equation 1. In fact, the cost of debt d, the return to levered equity e, the debt percentage %D and the equity percentage %E could vary each year.

Let $E_P\{FCF(1,1:2)\}$ be the expectation of the FCF for the two states of nature in year 1, with respect to P. The expected FCF is \$120.

$$E_P\{FCF(1,1:2)\} = p^U*FCF(1,1) + (1 - p^U)*FCF(1,2) = 50\%*190 + 50\%*50 = 120.00 \quad (3)$$

We assume that the return to unlevered equity ρ is 20%. Thus, with respect to year 0, the (present) value of the FCF, discounted with ρ, is \$100.

$$V^{Un}(0) = \frac{E_P\{FCF(1,1:2)\}}{1+\rho} = \frac{120}{1+20\%} = 100.00 \quad (4)$$

Equivalent probabilities (risk-neutral probabilities)

We can calculate the value of the FCF with respect to year 0 in another equivalent way. Rather than using the set of probabilities P, we define another set of equivalent probabilities Q = (q^U, (1 - q^U)). The set of equivalent probabilities Q (also known as risk-neutral probabilities), unlike the set of probabilities P, are very useful because they are appropriate (and convenient) for taking the expectation of any cash flow structure.[5]

To find the value of the FCF, we take the expectation of the FCF with respect to Q and discount with the risk-free rate r_f. Assume that the risk-free rate is 12%. We verify that the appropriate value of q^U is 44.29%.

Let $E_Q\{FCF(1,1:2)\}$ be the expectation of the FCF for the two states of nature in year 1, with respect to Q. The expected FCF is \$112.

$$E_Q\{FCF(1,1:2)\} = q^U*FCF(1,1) + (1 - q^U)*FCF(1,2) = 44.29\%*190 + (1 - 44.29\%)*50 = 112.0 \quad (5)$$

Thus, with respect to year 0, the (present) value of the FCF, discounted with r_f, is \$100.

$$V^{Un}(0) = \frac{E_Q\{FCF(1,1:2)\}}{1+r_f} = \frac{112}{1+12\%} = 100.00 \quad (6)$$

Debt financing

Next, we introduce debt financing. Let X(1) be the total payment (principal plus interest) for the debt in year 1 and let D(0) be the value of the debt in year 0. The maximum amount that can be repaid in year 1, with no risk, is equal to the value of FCF(1,2). That is,

$$D(0)*(1 + r_f) = FCF(1,2) \quad (7)$$

Solving for D(0), we obtain:

$$D(0) = \frac{FCF(1,2)}{1+r_f} = \frac{50}{1+12\%} = 44.64 \quad (8)$$

The critical value for the debt is \$44.64. If the value of the debt at the end of year 0 is higher than \$44.64, then there is a positive probability of default and the cost of debt will be higher than the risk-free rate of 12%.

For example, suppose the project "promises" to pay X(1) to the debt holder at the end of year 1, where the value of X(1) is \$60. At the end of year 0, what is the market value of the debt? Let CFD(i,j) be the cash flow to debt (CFD) in the jth state of nature in the ith year. If the up state of nature occurs, there will be no problem in making the payment of \$60 to the debt holder. However, if the down state of nature occurs, then the debt holder will simply receive \$50, which is the value of the FCF, and the equity holder will receive nothing.

$$CFD(1,1) = 60 \quad (9.1)$$

$$CFD(1,2) = 50 \quad (9.2)$$

To find the market value of the debt, we take the expectation of the CFD with the risk-neutral probabilities and discount with the risk-free rate.

Let $E_Q\{CFD(1,1:2)\}$ be the expectation of the CFD for the two states of nature in year 1, with respect to Q. The expected CFD is \$54.43.

5. There are two equivalent ways to calculate the value of an expected cash flow. First, we take the expectation of the cash flow with respect to a set of probabilities P and discount with the risk-adjusted discount rate. Second, we take the expectation of the cash flow with respect to Q, which is an equivalent set of probabilities, and discount with the risk-free rate. In terms of valuation, it is more convenient to take expectations with respect to the set of risk-neutral probabilities Q rather than the set of probabilities P.

   Since the risk-free rate is lower than the risk-adjusted rate, in the alternative method, the expectation of the cash flow has to be lowered accordingly by decreasing the probability for the up state of nature. Thus the value of qU, which is the equivalent probability for the up state of nature, is lower than the value of pU, which is the original probability for the up state of nature. Another way to think of the change in probabilities is as follows. Taking the expectation with respect to the equivalent set of probabilities Q, we "subtract" the risk premium from the cash flow to obtain the certainty equivalent. Consequently, we can discount the certainty equivalent with the risk-free discount rate.

$$E_Q\{CFD(1,1{:}2)\} = q^U{*}CFD(1,1) + (1 - q^U){*}CFD(1,2)$$
$$= 44.29\%{*}60 + (1 - 44.29\%){*}50 = 54.43 \qquad (10)$$

Thus, with respect to year 0, the (present) value of the FCF, discounted with r_f, is $48.60.

$$D(0) = \frac{E_Q\{CFD(1,1{:}2)\}}{1 + r_f} = \frac{54.43}{1 + 12\%} = 48.60 \qquad (11)$$

The value of the debt is 48.60% of the total value of $100. Below we show and explain the calculations for the promised and expected rate of return to debt.

Promised versus expected return on the debt

Since the debt is risky, we distinguish the promised rate of return on the debt d^{Prom} from the expected rate of return on the debt d^{Exp}. In the calculation of the WACC we use the expected rate of return on the debt rather than the promised rate of return.

Promised return to debt

The promised payment is X(1) and equal to $60. The promised rate of return is equal to 23.46%.

$$d^{Prom} = \frac{X}{D(0)} - 1 = \frac{60.00}{48.60} - 1 = 23.46\% \qquad (12)$$

In the down state of nature, the FCF of $50 is less than X(1) and the "promise" will not be fulfilled.

Expected return to debt

The relationship for the expected rate of return to debt is as follows.

$$\frac{E_Q\{CFD(1,1{:}2)\}}{1 + r_f} = \frac{E_P\{CFD(1,1{:}2)\}}{1 + d^{Exp}} \qquad (13)$$

The expectation of the CFD in year 1 with respect to Q and discounted with the risk-free rate is equal to the expectation of the CFD in year 1 with respect to P and discounted with the expected rate of return.

Let $E_P\{CFD(1,1{:}2)\}$ be the expected value of the CFD for the two states of nature in year 1, with respect to P. The expected CFD is $55.

$$E_P\{CFD(1,1{:}2)\} = p^U{*}CFD(1,1) + (1 - p^U){*}CFD(1,2)$$
$$= 50\%{*}60 + 50\%{*}50 = 55.00 \qquad (14)$$

Solving for the expected rate of return to debt in equation 13 and substituting the appropriate values, we obtain that the expected rate of return is 13.18%.

$$d^{Exp} = \frac{E_P\{CFD(1,1{:}2)\}{*}(1 + r_f) - 1}{E_Q\{CFD(1,1{:}2)\}}$$
$$= \frac{55{*}(1 + 0.12) - 1 = 13.18\%}{54.43} \qquad (15)$$

It can be shown that the expression for the expected rate of return to risky debt is as follows. See Tham & Wonder (2002) for details.

$$d^{Exp} = e - (e - r){*}\frac{V^{Un}(0)}{D(0)} \qquad (16)$$

Rate of return to equity

We can show that in the single period case, if the debt is risky, the rate of return to equity e is constant and the formula for the return to equity is as follows. See Tham and Wonder (2002) for details.

$$e = \frac{p^U{*}(1 + r_f)}{q^U} - 1$$
$$= \frac{50\%{*}(1 + 12\%)}{44.29\%} - 100\% = 26.44\% \qquad (17)$$

Weighted average cost of capital

We verify that the values for the cost of debt and return to equity are correct by showing that the WACC is equal to the return to unlevered equity ρ.

$$WACC = \%D{*}d + \%E{*}e$$
$$= 48.60\%{*}13.18\% + (1 - 48.60\%){*}26.44\%$$
$$= 20.00\% \qquad (18)$$

Expected return and promised return on risky debt

Thus, if the debt is risky, we can use equations 16 and 17 to estimate the cost of the debt as a function of the value of debt. For example, if the value of the debt in year 0 D(0) is $60, then the expected return on the debt is 15.71%

$$d^{Exp} = e - (e - \rho){*}\frac{V^{Un}(0)}{D(0)}$$
$$= 26.44\% - (26.44\% - 20\%){*}\frac{100}{60} = 15.71\% \qquad (19)$$

To borrow \$60 at the end of year 0, the project must promise to pay \$88.84 at the end of year 1, with a promised return of 48%. We verify that these numbers are correct.

$$E_Q\{CFD(1,1{:}2)\} = q^U{*}CFD(1,1) + (1 - q^U){*}CFD(1,2)$$
$$= 44.29\%{*}88.84 + (1 - 44.29\%){*}50 = 67.20 \quad (20)$$

Thus, with respect to year 0, the (present) value of the CFD, discounted with r_f, is \$60.

$$D(0) = \frac{E_Q\{CFD(1,1{:}2)\}}{1 + r_f} = \frac{67.20}{1 + 12\%} = 60.00 \quad (21)$$

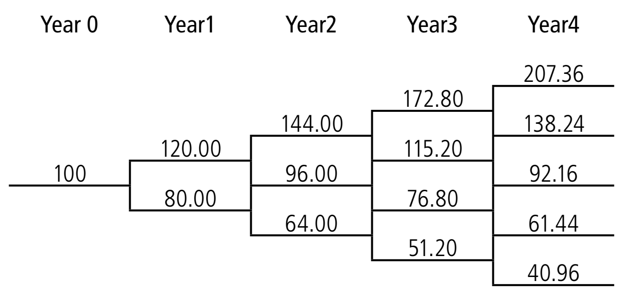
The expected rate of return to debt versus the value of debt in year 0

In the following graph, we plot the relationship between the expected rate of return to debt and equity versus the value of the debt in year 0. As expected, for values of debt less than or equal to \$44.64, the expected rate of return to debt is equal to the risk-free rate of 12%. As the value of debt increases above \$44.64, the expected rate of return to debt increases, at a decreasing rate.

For values of debt less than \$44.64, when the cost of debt is constant, the rate of return to equity increases, at an increasing rate. As the value of debt increases above \$44.64, the rate of return to equity is constant.

Section Two: multi-period binomial model

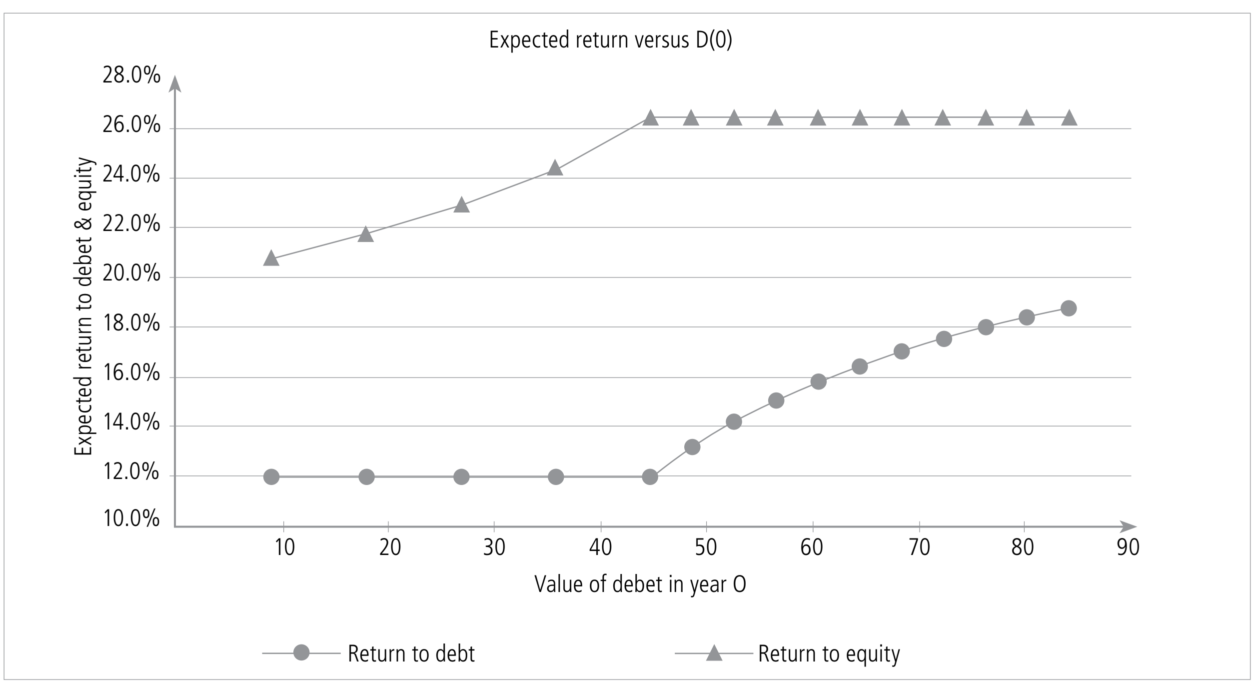
In this Section, we present a four period binomial model and use one-way tables to show the relationship between the cost of debt and the amount of debt. With a binomial model, we can see clearly the impact of the passage of time on the expected return to debt and equity. For convenience we use a recombining tree to represent the (present) value of the FCF rather than the FCF process.[6] Let $V^{Un}(i,j)$ be the (present) value of the FCF in the jth state of nature in the ith year. We assume that $V^{Un}(i,j)$ either increases or decreases by 20%.



Graph 2: Process for the unlevered value VUn(i,j)

There are no free cash flows in years 1 to 3. The only cash flows occur in year 4.

6. To be correct, we should model the cash flow process rather than the value process. Modeling the cash flow process is more difficult.



Graph 1: Relationship between the expected rates of return and the value of debt D(0)

$$FCF(i,j) = 0 \quad \text{for all } i \leq 3 \qquad (22.1)$$

$$FCF(i,j) = V^{Un}(i,j) \quad \text{for } i = 4 \qquad (22.2)$$

For example, under the third state of nature in year 4, the equity holder receives $92.16.

The reader can verify that the correct values of the parameters for the above unlevered value process are as follows.

$$P = \{p^U, (1 - p^U)\} = \{70\%, 30\%\} \qquad (23.1)$$

$$Q = \{q^U, (1 - q^U)\} = \{62.5\%, 37.5\%\} \qquad (23.2)$$

$$\text{Unlevered return } \rho = 8\% \qquad (23.3)$$

$$\text{Risk-free return } r_f = 5\% \qquad (23.4)$$

Debt financing (1)

Let X(4) be the payment to the debt holder in year 4. There are no cash flows to debt in years 1 to 3. If the value of X(4) is less than or equal to $40.96, which is the FCF under the fifth state of nature in year 4, then the debt is risk-free and the return to debt is equal to the risk-free rate. If the value of X(4) is higher than $40.96, then the debt is risky because there is a positive probability that the debt will not be fully repaid and the return to debt will be higher than the risk-free rate.

Suppose the value of X(4) is $60, which is between the values of the FCF under the fourth and fifth states of nature. With respect to year 0, what is the value of the debt D(0)? To find the value of the debt, first we calculate the CFE and then the CFD.

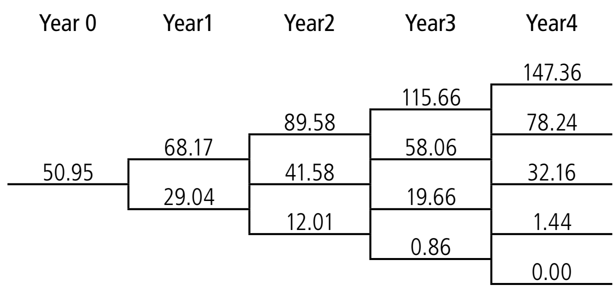
In year 4, under each state of nature, the cash flow to equity is as follows.

$$CFE(4,j) = \max\{[FCF(4,j) - X(4)], 0\} \qquad (24)$$

If the FCF is greater than the promised payment for debt, then the CFE is the difference between FCF and the payment. Otherwise, the equity holder receives zero. The cash flow to debt is simply the difference between the FCF and the CFE.

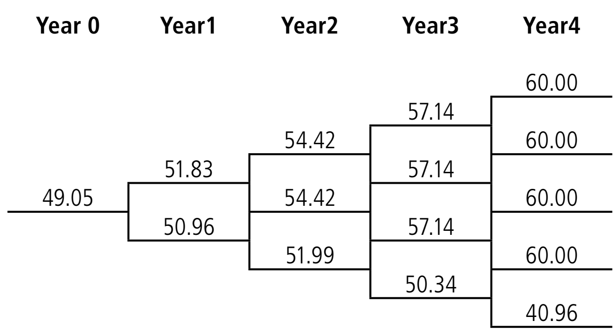
Below, we show the process for the value of equity $E^L(i,j)$ and the value of debt D(i,j). Under the first four states of nature in year 4, the FCF is sufficient to pay the debt holder. In the fifth state of nature, the FCF is insufficient to pay the debt holder.

Under the first four states of nature in year 4, the debt holder receives the promised amount X(4) and in the fifth state of nature, the debt holder receives the FCF of $40.96.



Graph 3: Process for the value of (levered) equity $E^L(i,j)$ with X(4) = 60

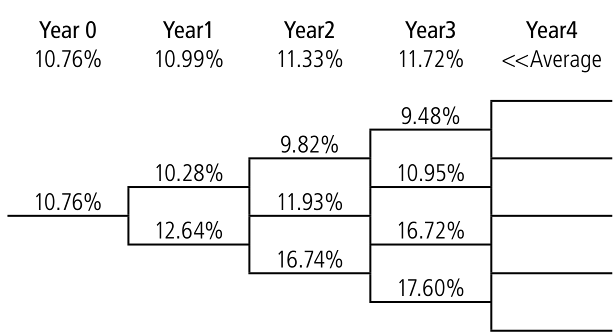
To find the value at any node in the value trees for the equity and debt, we take the expectation with respect to the set of risk-neutral probabilities and discount with the risk-free rate rf.



Graph 4: Process for the value of debt D(i,j) with X(4) = 60

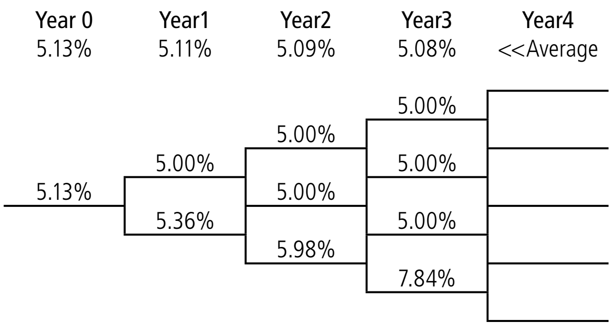
Return to equity and return to debt

Next we show the returns to equity and debt at each node of the value trees. The average (expected) return to equity increases from 10.76% in year 0 to 11.72% in year 3.



Graph 5: Return to levered equity e(i,j) with X(4) = 60

The average (expected) return to debt decreases from 5.13% in year0 to 5.08% in year 3.



Graph 6: Return to debt D(i,j) with X(4) = 60

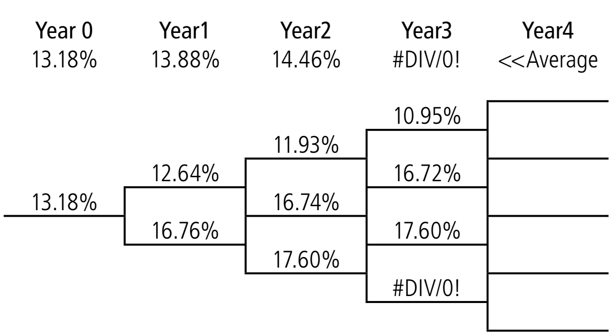
We verify that the WACC in year 0, calculated with the expected return to equity and debt, is equal to the return to unlevered equity ρ.

$$\text{WACC}(0) = \%D*d + \%E*e$$
$$= 49.05\%*5.13\% + (1 - 49.05\%)*10.76\%$$
$$= 8.00\% \qquad (25)$$

For the convenience of the reader, in appendix A, we present an example of a calculation for the return to debt at the third node in year 2 d(2,3).

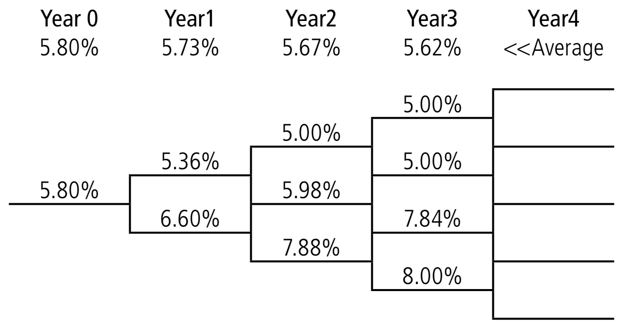
Debt financing (2)

Next we consider a higher level of debt financing. Suppose the value of X(4) is $90, which is between the values of the FCF under the third and fourth states of nature. With respect to year 0, what is the value of the debt D(0)? Again, we can construct the tree processes for the return to equity and the return to debt. Under the fourth state of nature in year 3, the return to equity is undefined because CFE is less for both the fourth and fifth states of nature in year 4.



Graph 7: Return to levered equity e(i,j) with X(4) = 90

With the increase in X(4) from $60 to $90, in year 0, the expected return to equity has increased from 10.76% to 13.18% and the expected return to debt has increased from 5.13% to 5.80%.

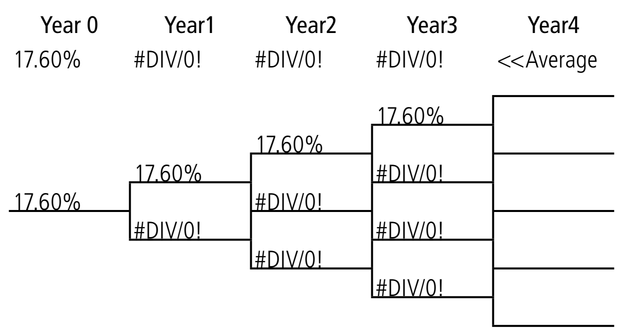


Graph 8: Return to debt D(i,j) with X(4) = 90

With X(4) equal to $90, the reader can verify that the value of the debt in year 0 is $70.15.

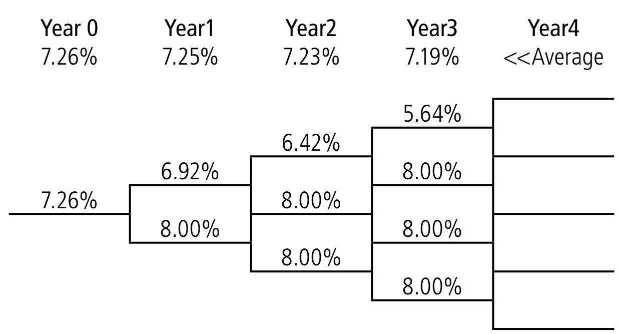
Debt financing (3)

Next, we examine a very high level of debt and set X(4) equal to $150, which is between the values of the FCF under the first and second states of nature.



Graph 9: Return to levered equity e(i,j) with X(4) = 150

With such a high level of debt, the CFE will be positive in the first state of nature in year 4 and zero in all the other states. Now the expected return to equity is constant at 17.60%.



Graph 10: Return to debt D(i,j) with X(4) = 150

For the debt holder, the debt will only be repaid in full under the first state of nature in year 4. Under all the other states of nature in year 4, the debt holder will receive **less** than the promised amount. The expected return to debt is 7.26%.

The expected rate of return to debt and equity versus the value of debt in year 0

In the following graph, we plot the relationship between the expected rate of return to debt and equity versus the value of the debt in year 0.

Note that when the value of X(4) is between the values of the FCF under the first and second states of nature, as discussed previously, the expected return to equity is constant at 17.60%.

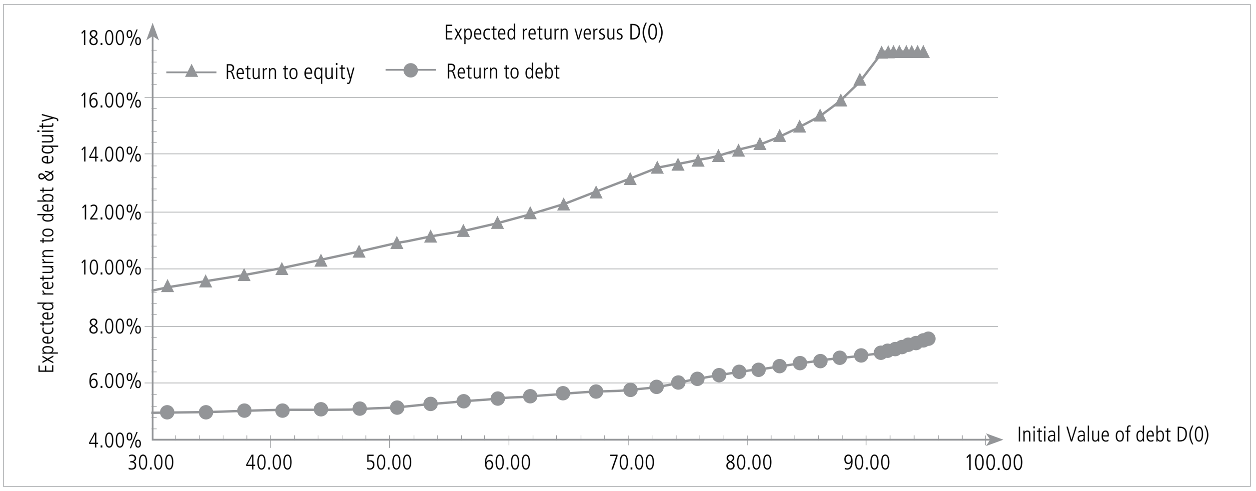
Using the binomial model and a one-way table, we can estimate the expected return to equity and risky debt as a function of the value of debt in year zero.[7]

the value of the debt at the subsequent two nodes with respect to P, discounted with d(2,3).

$$\frac{E_Q\{D(3,3:4)\}}{1 + r_f} = \frac{E_P\{D(3,3:4)\}}{1 + d(2,3)} \tag{A1}$$

$$E_P\{D(3,3:4)\} = p^U*D(3,3) + (1 - p^U)*D(3,4)$$
$$= 70\%*57.14 + 30\%*50.34 = 55.10 \tag{A2}$$

$$E_Q\{D(3,3:4)\} = q^U*D(3,3) + (1 - q^U)*D(3,4)$$
$$= 62.5\%*57.14 + 37.5\%*50.34 = 54.59 \tag{A3}$$



Graph 11: Relationship between the expected rates of return and the value of debt D(0)

Conclusion

We used simple binomial models to illustrate the WACC with risky debt and no taxes. In the single period case, when the debt is risk-free, the cost of debt is constant and the return to equity is a function of the debt-equity ratio. When the debt is risky, the return to equity is constant and the expected return to debt is a function of the value of debt. In the multi-period case, we use one-tables to show the relationship between the rates of return to equity and debt as functions of the value of debt in year 0.

APPENDIX A: Calculating the expected return on debt

The expectation of the value of the debt at the subsequent two nodes in year 3 with respect to Q, discounted with the risk-free rate rf is equal to the expectation of

Substituting the appropriate values, we obtain that the expected rate of return on the debt is 5.98%.

$$D(2,3) = \frac{E_P\{CFD(1,1:2)\}*(1 + r_f) - 1}{E_Q\{CFD(1,1:2)\}}$$
$$= \frac{55.10*(1 + 5\%) - 1}{54.59} = 5.98\% \tag{A4}$$

References

THAM, J & WONDER, N. (2001) “The Non-conventional WACC with Risky Debt and Risky Tax Shield.” Working Paper. Available on the Social Science Research Network (SSRN)

7. It is interesting that numerically we can use the formula for the expected cost of debt for the single period given in equation 16 to generate the expected cost of debt in the multi-period example. However, we have not attempted to show rigorously that the formula in equation 16 applies in general.