# Mathematical Understanding and the Role of Counterexamples and Pathologies: A Case Study in Mathematical Analysis[1,2]

## La comprensión matemática y el papel de los contraejemplos y las patologías: un estudio de caso de análisis matemático

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**ABSTRACT**

Pathological objects play an important role in mathematical understanding even though there is no precise definition of them. What is a pathological object? What makes a mathematical object pathological? The aim of this paper is to give a partial response to these questions from the standpoint of mathematical analysis in the nineteenth century and the first quarter of the twentieth century. It will be briefly described how the notion of function changed dramatically in the nineteenth century and it will be studied how this change brought on important philosophical consequences for the subject, which lead to the conclusion that the notion of pathology relies upon certain properties that occur only in a *few* instances.

**Keywords**: Philosophy of mathematics, pathological objects, continuous functions.

**Resumen**

Los objetos patológicos juegan un papel importante en la comprensión matemática a pesar de que no hay una definición precisa de lo que son. ¿Qué es un objeto patológico? ¿Qué hace que un objeto matemático sea patológico? El objetivo de este artículo es dar una respuesta parcial a estas preguntas desde el punto de vista del análisis matemático del siglo diecinueve y el primer cuarto del siglo veinte. Se describirá brevemente el cambio dramático que tuvo la noción de función en el siglo diecinueve, y se estudiará el modo en que este cambio trajo consigo consecuencias filosóficas importantes para la materia, que llevan a la conclusión de que la noción de patología descansa sobre ciertas propiedades que ocurren únicamente en unas pocas instancias.

**Palabras clave**: Filosofía de las matemáticas, objetos patológicos, funciones continuas.

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## 1. Introduction

Mathematical understanding is a subject that has been studied from several points of view. The goal of this paper is to study how mathematics is understood within mathematics. A highly relevant question in this process is how and when an object becomes a mathematical object. This makes it possible to analyze the origin and context in which different branches of mathematics arise or expand and it is interesting to question if this happens in the same manner for all mathematical branches. I am mainly interested in studying the role that pathological objects play in this process and how they are linked to the phenomenon of understanding within mathematics. To do this I will focus on a particular case, that of continuous non-differentiable functions.

Since its consolidation as a mathematical branch, Mathematical Analysis has dealt with the study of functions and function spaces. This became clear as early as 1748 with the publication of Euler's *Introductio in Analysin Infinitorum*, where he claimed that the whole of Mathematical Analysis is dedicated to the study of functions. Now, in order to be able to study these objects in the terms stipulated earlier it is necessary to understand what a function actually is. I do not intend to give a full account of the development of this concept, but it is essential for the present study to mention at least a few key points and so will be done in the next section.

Once this has been done, it will be shown how functions that arose as apparent counterexmaples eventually turned out not to be so, since a vast majority --in a precise mathematical sense-- of functions behave in the same way. This fact could not have been known *a priori*, since the evolution of the understanding of what a function actually is played a crucial role in obtaining these results; and this evolution, in turn, depended on the existence of the counterexamples.

## 2. Functions, continuity and differentiability: a very brief account

Throughout mathematical history the problem of classification of both mathematical objects and problems has been of constant interest, and the study of functions is no different in this respect.

The first published definition of function was given by Johann Bernoulli (106): "One calls function of a variable magnitude a quantity compounded in any way of this variable quantity and of constants.”,[6] and it was only very slightly modified by Euler in his Introductio (1748 4): “A function of a variable quantity is an analytic expression compounded, in whichever way, of that same variable quantity and numbers or constant quantities.”[7] This idea of a function as an analytic expression was largely responsible for the idea that functions as such were continuous and differentiable, even though both of these notions appeared at a later date.

For Euler, the classification of functions had to do with the type of analytic expression involved, and the continuous/non-continuous distinction was reserved for curves. The term ‘discontinuous’ appeared as such, in his later work, applied to functions, but all of the functions considered by him were continuous in the modern sense. It would appear that even when a more general definition of function was considered, as in the *Institutiones calculi differentialis* (1755)[8], all functions considered fell within the scope of the older definition.

The fact that all of the examples and individual functions studied were continuous in the modern sense set the standard to which the behaviour of functions in general was held. That is, at the beginning of the nineteenth century, continuity and differentiability were treated as necessary properties of functions, in the sense that any function would have been thought to be continuous and differentiable at *most* of its points.

## 3. Pathological functions

The term ‘pathological function’ is a widely used and understood expression which usually refers to continuous functions that are non-differentiable in a set of non-isolated points. However, the concept is actually much broader; it can refer to any function that behaves atypically bad or counterintuitively. This

6 “On appelle fonction d’une grandeur variable une quantité composée de quelque manière que ce soit de cette grandeur variable et de constants.” (Translations from French by Yenni Castro).

7 “Functio quantitas variabilis, est expressio analytica quomodocunque composita ex illa quantitate variabili, & numeris seu quantitatibus constantibus.” (Translations from Latin and Italian by Miguel Araneda).

8 In the preface Euler states that: “Those quantities that depend on others in such a way that, if the latter change they also change, are called functions of the latter; this denomination is widely patent and it comprises all the ways in which one quantity may be determined by others. Therefore, if x stands for a variable quantity, all quantities which depend on x in any way, or are determined by it, are called functions of x.”
“Quae autem quantitates hoc modo ab aliis pendent, ut his mutatis etiam ipsae mutationes subeant, eae harum functiones appellarie solent; quae denominatio latissime patet atque omnes modos, quibus una qunatitas per alias determinari potest, in se complectitur. Si igitur denotet quantitatem variabilem, omnes quantitates, quae utcunque ab pendent seu per eam determinantur, eius functiones vocantur.”

is the reason why the study of these types of functions holds a strong link to mathematical understanding. It allows us to see what a mathematician would have expected to happen in her everyday practice and what actually holds.

In the strict sense mentioned above, the first pathological function to be published was presented publicly by Weierstrass in front of the Berlin Academy on July 18, 1872. It was published by Paul du Bois-Reymond in 1875 as follows: $f(x)=\sum_{k}^{\infty} k=0\cos\left(b^{k}\pi x\right)$ with $0>a>1, ab>1+\frac{3\pi}{2}$ and $b>1$ an odd integer. Since it was the first function of this sort to be published, it is regarded by many as the first known function of this type. However, it is important to note that throughout the nineteenth century there were many other examples of pathological functions, including one studied by Bolzano as early as 1830.

The function presented by Bolzano in his *Functionenlehre*, which unfortunately was not published until 1930, is probably the first example of a continuous nowhere differentiable function. Unlike many of the other constructions of nowhere differentiable functions that were to follow, Bolzano's function rests upon a geometrical construction instead of a series approach. The function was first presented as an example of a function that is continuous in an interval but is not monotone in any subinterval. Bolzano then stated that "the function $Fx$ with which the rising and falling alternates so frequently that for no value of $x$ is there an $\omega$ small enough to be able to assert that $Fx$ always increases or always decreases within $x$ and $x\pm\omega$, gives us a proof that a function can even be continuous and yet have no derivative."[9]

Given the mathematical context in which Bolzano carried out his work, Magdalena Hykšová states: "Already the fact that it occurred to Bolzano at all that such a function might exist, deserves our respect. The fact that he actually succeeded in its construction, is even more admirable" (72). I agree with Hykšová given that, as mentioned before, even a couple of decades prior to Bolzano's work the concepts of function, continuous function and differentiable function seemed to be so intertwined that they were practically indistinguishable, and Bolzano was well aware of this fact.

Immediately after his proof that the function $Fx$ was non-differentiable, Bolzano included a note in which he acknowledges that his theorem "contra-

9 Translations into English are taken from (Russ 2004 507). Bolzano actually proved that the function is non-differentiable on what we would call today a 'dense set'. It was Jarník in 1922 who proved that it is nowhere differentiable.

dicts to a certain extent what Lagrange and many others sometimes explicitly claim, and sometimes just tacitly assume: that every function, with at most the exception of some isolated values of its variable, but in all other cases, has a derivative" (508). He then goes on to say that the word 'function' was taken in a much narrower sense by the mathematicians he is referring to:

> they understand by it only such numbers, dependent on another number which can be expressed by one of the seven signs: $a+x, a-x, ax\frac{a}{x}, x^n, a^x, \log x$ or by a combination of several of these. Now what they claim holds of such functions especially as with some of these signs it is already in the meaning of them, that they should denote numbers that vary only by the law of continuity, or always have a derivative. But since I believe that a much wider concept must be associated with the word function then it will be necessary to allow of functions that they not only have no derivative, but they may even break the law of continuity… (Russ 508).

Note that Bolzano's insight does not only depend on considering a wider class of functions (whose existence he was then obviously require to show); for as he himself observes, even in 1830 other mathematicians that shared his view of functions in a wider sense were still inclined to believe that every function had a derivative, provided some isolated values were excluded. Such was the case of Galois,[10] who in 1830 in the *Annales de Mathématiques Pures et Appliquées* (T. 21 182) published the following theorem:

> Theorem: Let $Fx$ and $fx$ be two arbitrarily given functions; one will have, whatever $x$ and $h$ are, $\frac{F(x+h)-Fx}{f(x+h)-fx} = \varphi(k)$, $\varphi$ being a certain function and k an intermediary quantity between x and $x+h$.[11]

Galois ended his proof by saying: ". . . which demonstrates, a priori, the existence of derived functions."[12] The fact that the quotient was assumed to be a number $\varphi(k)$, does not only imply that this function depends on $x$ and $h$ but also that it depends on the nature of $f$ and $F$. This last assumption

10 His name in the *Annales de Mathématiques* appears as 'Galais'.

11 Théorème : Soient $Fx$ et fx deux fonctions quelconques données ; on aura, quels que soient $x$ et $h$, $\left(F(x+h)-Fx\right)/\left(f(x+h)-fx\right)=\varphi(k)$, $\varphi$ étant une fonction déterminée, et $k$ une quantité intermédiaire entre $x$ et $x+h$.

12 "ce qui démontre, à priori, l'existence des fonctions dérivées" (184)

was the one Galois did not considered, thus rendering his proof only valid if $f$ and $F$ were continuous. It is not intended here to point out an error in Galois' reasoning, for I do not believe this to be the case. I believe that the process through which functions and continuous functions were to become two clearly defined and separated concepts had only just begun, and the true nature of their connection had yet to be fully understood. In this context, Bolzano's work is truly admirable.

It was not until the end of the nineteenth century that the matter changed dramatically, not, however, without some resistance. In 1922 Lebesgue, in his *Notice sur les travaux scientifiques* (13), remembered that:

> In 1899, I had given to Mr. Picard a note about non-ruled surfaces that can be projected on a plane; for a moment Hermite wanted to oppose to its insertion in the Proceedings of the Academy of Sciences . . . It was around the time when he wrote to Stieltjes: «I turn away with fear and horror from that pitiful plague of functions without derivatives».[13]

A similar point of view was held by Poincaré who in 1908, in Science *et méthode*, wrote:

Sometimes logic gives rise to monsters. For half a century there has emerged a multitude of strange functions that seem to make an effort to resemble as least as possible to the honest functions that serve some purpose. More than continuity, or perhaps continuity, but no derivatives, etc. Much more, from a logical point of view, these strange functions are the most general ones; those that are found without having looked for them only appear as a particular case. Formerly, when one invented a new function, it was with a practical goal in view; nowdays, they are invented specifically for making the reasonings of our forefathers fall into error, and nothing else will come out it (132).[14]

It would seem that the resistance to these functions came from their apparent lack of necessity. I believe that Poincaré was ultimately wrong in this last

13 "En 1899, j'avais remis à M. Picard une note sur les surfaces non réglées applicables sur le plan; Hermite voulut un instant s'opposer à son insertion dans les Comptes Rendus de l'Académie [...] c'était à peu près l'époque où il écrivait à Stieltjes: «Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions qui n'ont pas de dérivées»" (132).

14 "La logique parfois engendre de monstres. Depuis un demi-siècle on a vu surgir une foule de fonctions bizarres qui semblent s'efforcer de ressembler aussi peu que possible aux honnêtes fonctions qui servent à quelque chose. Plus de continuité, ou bien de la continuité, mais pas de dérivées, etc. Bien plus, au point de vue logique, ce sont ces fonctions étranges qui sont les plus générales, celles qu'on rencontre sans les avoir cherchées n'apparaissent plus que comme un cas particulier [...] Autrefois, quand on inventait une fonction nouvelle, c'était en vue de quelque but pratique; aujourd'hui, on les invente tout exprès pour mettre en défaut les raisonnements de nos pères, et on n'en tirera jamais que cela." (142)

claim, as something truly important did emerge from this *foule* of functions: the understanding of what a function actually is; not as a single object, but as an object in a class. It was the surge of functions of this type that allowed a further classification of functions as had been sought since Euler's time.

As a proof of the change regarding functions that occurred within the mathematical community, we can mention another function proposed by Charles Cellérier that was only published posthumously (1890). However –given the aim of this paper– more important than the function itself, which is $\varphi(x) = \sum_{n=1}^{\infty} \frac{\sin(a^n x)}{a^n}$, is the footnote that appears in the paper. It reads: "This memoir was found in the documents of Mr. Cellérier, professor at Geneva, who died last year. It is completely of his own writing . . . The author wrote in the paper that contains it the following superscription: «Very important, and, I think, new. –Correct writing. It can be published as it is.»" [15] This note reflects the novelty of the subject at hand and the importance attributed to it by its author.

By the early twentieth century many more examples of such functions arose (as Poincaré signaled in his texts) and it is important to note that, even though the goal of this paper is to trace the development of continuous non-differentiable functions from isolated pathologies to a very large class of functions, many other functions, with different pathologies, arose.[16] I believe that these other types of functions made it very clear that it was the concept of function as a whole which had to be broadened, not just the concept of continuous functions.

One clear example of this is the paper published by Peano in 1890, Sur une nouvelle courbe continue qui remplit toute une aire plane. In this note Peano presents a continuous curve whose coordinates in the plane are given by continuous functions that cannot be enclosed in an infinitely small area.

Once it was clear that many such pathological functions existed, one of the issues was to determine through which methods could they be obtained. For instance,

15 "Ce Mémoire a été trouvé dans les papiers de M. Cellérier, professeur à Genève, mort l'année dernière. Il est entièrement écrit de sa main […] l'auteur a mis sur la feuille qui le renfermait la suscription que voici: «Très important, et, je crois, nouveau. – Rédaction correcte. Peut être publié tel quel»

16 Examples of these include space filling curves, non-continuous functions that satisfy the intermediate value property, continuous functions that are not absolutely continuous like the Cantor ternary function, etc.

in 1904, Koch presented a paper (which was republished with a few additions two years later) called "Sur une courbe continue sans tangente, obtenue par une construction géometrique élémentaire", in which he describes a curve of infinite length with no tangents, the famous Koch snowflake. Koch writes:

> Until Weierstrass constructed a continuous function not differentiable at any value of its argument, it was widely believed in the scientific community that every continuous curve has a well determined tangent [...] Even though the example of Weierstrass has corrected this misconception once and for all, it seems to me that his example is not satisfactory from the geometrical point of view since the function is defined by an analytic expression that hides the geometrical nature of the corresponding curve and so from this point of view one does not see why the curve has no tangent.[17]

It is clear from Koch's paper that Bolzano's function was not known to the community, since the construction given by Bolzano of his non-differentiable continuous function, as mentioned earlier, relies on a geometric construction. Another issue studied, once the countless examples of pathological functions began to flow, was how general these functions actually were.

## 4. From pathological counterexamples to the generality theorem

So far, we have given a few examples of the many individual pathological functions that arose in the late nineteenth and early twentieth centuries (even though there were parameters present in some of them). These examples led to many questions and a lot of research was carried out in several directions. Each example was studied individually, new examples were created, and even different definitions of differentiation were proposed. However, it is probably fair to say that it was Ulisse Dini who took the first steps towards a general theory. In 1877 he stated a general existence theorem, whose proof appeared in 1878 together with his systematized research on the subject: "The theorem that follows provides new and infinite classes of functions that, however finite, continuous and provided of a complex analytic expression –that is usually also very simple– do not admit a determined and finite derivative. Among

17 The English translation is taken from (Edgar 25).

these functions it is included as a particular case the one studied by Mr. du Bois-Reymond."[18]

Dini made a point to show that the class of functions described by his theorem is infinite and in fact contains Weierstrass' function. He even proved that under additional assumptions the functions proposed do not even have infinite derivatives at any point.

This proof of the fact that there are infinite classes of continuous functions that are non-differentiable led to a work by Darboux, where he presented a general method for creating such classes of functions, and to a search for different conditions for differentiability. Then, the next logical step was the generalization of the concept of differentiation. The discovery of these *monsters* not only proved to be interesting in its own right, but allowed for a vast branch of mathematics to develop further.

In 1929, Hugo Steinhaus concluded his paper, by stating three unresolved problems, the third of which was: to determine the category[19] of the set of continuous nowhere differentiable functions in the space of all continuous functions. In 1931, both Banach and Mazurkiewicz published papers in the third volume of *Studia Mathematica* that gave a response to this problem using slightly different methods. They both proved that the set of functions that do not have a finite right derivative is of the second category in the space of all continuous functions. That is, the set of functions that was originally thought to be comprised of all functions at the beginning of the nineteenth century turned out to be negligibly small in comparison with the set of all continuous functions.

It would then appear that the concept of function had evolved so much as to completely change, and practically reverse, the meaning of examples and counterexamples within the theory, rendering the term 'pathological function' obsolete as a way of describing the functions that were first classified as such, but incredibly fruitful in enabling a much better comprehension and understanding of the subject.

I would like to close with the following quote taken from Medvedev (233): "The class of continuous nowhere differentiable functions turned out to be

18 "Il seguente teorema dà nuove e infinite classi di funzioni che, per quanto finite e continue e dotate di una espressione analitica che spesso è anche molto semplice, non ammettono mai una derivate determinate e finite. Fra queste funzioni è compresa come caso particolare quella studiate dal signor du Bois-Reymond."

19 In the sense of Baire Category as introduced by René Baire in 1899 in his doctoral thesis.

immeasurably richer than the class of differentiable functions and it was rather functions of the latter type that were "pathological"."

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