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Quantum trans-statistics from an ontology of properties*

Trans-estadística cuántica desde una ontología de propiedades*

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ABSTRACT

In recent years, the bosonic behavior that a many-fermion system can exhibit has raised interest among physicists. In this paper, an approach based on tensor product structures is taken and an ontology of properties is assumed to argue for the relativity of the notion of statistical identity and for a realistic interpretation of trans-statistical behavior.

Keywords: philosophy of physics; virtual particles; composite bosons; ontology of properties; tensor product structure.

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RESUMEN

En los últimos años, el comportamiento bosónico que un sistema de fermiones puede exhibir ha despertado el interés de los físicos. En este trabajo, se adopta un enfoque basado en estructuras producto tensorial y se asume una ontología de propiedades para argumentar en favor de la relatividad de la noción de identidad estadística y en favor de una interpretación realista del comportamiento trans-estadístico.

Palabras clave: filosofía de la física; partículas virtuales; bosones compuestos; estructura producto tensorial; ontología de propiedades.

1. Introduction

According to quantum mechanics, systems composed of identical particles are either fermions or bosons in a mutually exclusive manner. However, under certain circumstances, pairs of fermions can behave as composite bosons, called cobosons. Generally, in the realm of physics, this behavior is not considered to be realistically interpretable but is merely a description based on idealized, and therefore approximate, models. This article defends the idea that the bosonic behavior that a system of fermions can exhibit can be realistically interpreted if certain ontological commitments are assumed. To do this, it will use: (1) a simple toy model that employs different partitions of Hilbert space, also called tensor product structures (TPS), from which it is argued that the statistical identity (fermionic or bosonic) of a system composed of identical particles is relative to the TPS; and (2) an ontology of properties where quantum systems are seen as clusters of possible properties, without identity conditions typical of individual objects. The structure of the article is as follows: Section 2 introduces the reader to a quantum feature called indistinguishability, which determines the identity and behavior upon permutations of quantum systems. Section 3 addresses the phenomenon of trans-statistical behavior and its conventional interpretation in the physical realm. Section 4 briefly presents some results of the approach based on TPSs that several authors have developed in recent years, an approach in which this proposal seeks to be inscribed. Section 5 proposes a version in Hilbert space formalism of the toy model, suggesting the relativity of the notion of statistical identity with respect to the TPS. Section 6 provides an overview of the ontological challenges posed by quantum mechanics and introduces the ontology of possible properties, inspired by algebraic formalism. Finally, Section 7 presents an algebraic formalism version of the toy model from Section 5, showing more clearly the relevance of a property ontology to explain trans-statistical behavior realistically.

2. QUANTUM INDISTINGUISHABILITY AND STATISTICAL BEHAVIOR

The statistical behavior of quantum systems, whose subsystems possess identical properties, differs significantly from that of their classical counterparts. For reasons that will be explained below, this quantum behavior is based on the premise that quantum subsystems of the same type are indistinguishable. Statistical mechanics arises to explain the behavior of complex systems whose macroscopic properties are derived from the states of their subsystems. Take, for example, a volume of gas at a certain temperature, which is related to the kinetic energy of its molecules. Since it is unfeasible to obtain information about the individual state of each subsystem, statistical methods are used. These methods assume that all possible distributions of the subsystems among the different states have the same probability. In the following example, these distributions of subsystems among possible states will be called 'complexions'. Consider two subsystems S_A and S_B, and two posible states $|\Psi_1\rangle$ and $|\Psi_2\rangle$. If the subsystems are classical, the following complexions are obtained (Fortin & López 2016):

(1) S_A and S_B both are found in the state $|\Psi_1\rangle$.

(2) S_A is found in the state $|\Psi_1\rangle$ and S_B in the state $|\Psi_2\rangle$.

(3) S_B is found in the state $|\Psi_1\rangle$ and S_A in the state $|\Psi_2\rangle$.

(4) S_A and S_B both are found in the state $|\Psi_2\rangle$.

It's important to note that when taking complexion (2) and permuting the subsystems between the states (or, equivalently, permuting the states between the subsystems), one obtains complexion (3). The statistics that arise when considering that permutations of subsystems result in different complexions is known as the Maxwell-Boltzmann distribution and is used in the classical context.

Now consider the case of quantum subsystems, in particular, particles with integer spin or bosons. The following complexions are obtained:

(1) Both systems are in the state $|\Psi_1\rangle$.

(2) One system is in the state $|\Psi_1\rangle$ and the other in the state $|\Psi_2\rangle$.

(3) Both systems are in the state $|\Psi_2\rangle$.

It should be noted that for quantum subsystems, it doesn't make sense to retain the labels S_A and S_B. This is because, unlike the classical case, when taking complexion (2) and permuting subsystems, one doesn't obtain a distinct complexion that needs to be statistically accounted for. Contrary to what one might expect, permuting quantum subsystems of the same type has no observational consequences. This peculiarity of quantum subsystems is reflected in what is called the indistinguishability postulate IP_{st}) (the standard specification is added because a version of IP centered on observables rather than states will be introduced later):

> IP_{st}: If the vector $|\psi\rangle$ represents the state of the composite system whose subsystems are indistinguishable, then the expected

value of any observable represented by an operator O must be the same for $|\psi\rangle$ and for any permutation $|\psi'\rangle$

$$\text{PI}_{\text{st}}\text{:}\ \langle\psi'|O|\psi'\rangle = \langle\psi|O|\psi\rangle \ \text{ siendo } \ |\psi'\rangle = P|\psi\rangle$$ MERGEFORMAT

Where P is a generic permutation operator (see Butterfield 1993). This postulate led some of the founding fathers of quantum mechanics, such as Born and Heisenberg, to consider that quantum systems might behave as non-individual entities. Indeed, among quantum systems, there can be a mere numerical difference, even when all their properties are indistinguishable. If there were numerical identity between them (i.e., if they were the same object), only two complexions would have been counted in the previous example. If the systems had at least some distinguishable properties, we should have counted four complexions, as it is the case in the classical scenario. However, the statistical law applicable to bosons leads us to account for only three complexions in our example. It is imperative to accept that, in the quantum context, indistinguishable systems can be numerically distinct, resulting in a violation of Leibniz's principle of the identity of indiscernibles. This principle, formulated in classical metaphysics, serves as a criterion for numerical identity for classical systems that possess identity conditions defining them as individual objects.

The quantum statistics mentioned in the previous example is known as Bose-Einstein statistics. Hence, integer spin particles that follow this statistical law are called "bosons". Correspondingly, we will say that indistinguishable systems governed by this law have bosonic statistical identity (using the notion of identity here in terms of qualitative identity). However, it is essential to note that this is not the only statistics applicable to quantum systems. Half-integer spin particles, or fermions, are governed by the Pauli exclusion principle, which prohibits two or more particles from coexisting in the same state. For these particles, the only possible complexion in our example of a system composed of two subsystems and two states is:

(1) One system is in the state $|\Psi_1\rangle$ and the other in the state $|\Psi_2\rangle$.

Fermions obey the statistics known as Fermi-Dirac, from which they derive their name. We will say that indistinguishable systems that adhere to this statistical law have fermionic statistical identity (in the sense of qualitative identity). The differentiation between the two types of statistical identity (fermionic and bosonic) in the quantum context is due to the fact that IPst is fulfilled only in symmetric states $|\psi_S\rangle$ or antisymmetric $|\psi_A\rangle$ regarding the permutation operators. Both types of states are eigenstates of the possible permutation operators, with eigenvalues (1) and (-1) respectively. That is:

$$\begin{aligned} P|\psi_S\rangle &= |\psi_S\rangle \\ P|\psi_A\rangle &= -|\psi_A\rangle \end{aligned}$$ MERGEFORMAT

Due to the nature of the formalism, the previously mentioned exclusion principle is clarified, introduced initially to explain the distribution of electrons among different energy levels in atomic structure. Because of this, an additional postulate was incorporated into the formalism of quantum mechanics: the symmetrization postulate (SP). This postulate represents an *ad hoc* restriction on the set of possible states that the formalism admits for systems with indistinguishable subsystems, ensuring in these, the satisfaction of IP_{st}. SP can be formulated as follows (Fortin & Lombardi 2021):

> SP: A multi-particle system of identical particles must be represented by a quantum state that is either fully symmetric (bosons) or fully antisymmetric (fermions), where the symmetry or antisymmetry is defined in terms of the permutation operators P.

$$|\psi'\rangle = P|\psi\rangle = \pm|\psi\rangle$$ MERGEFORMAT

To obtain symmetric $|\psi_S\rangle$ or antisymmetric $|\psi_A\rangle$ states, symmetrization or antisymmetrization A operators must be applied respectively to a generic state $|\psi\rangle$

$$\begin{aligned} S|\psi\rangle &= |\psi_S\rangle \\ A|\psi\rangle &= |\psi_A\rangle \end{aligned}$$

MERGEFORMAT

Based on what has been presented so far, it seems logical to interpret that the statistical identity of a composite system is predetermined in an absolute manner at a more fundamental level than that directly described by the formalism of quantum mechanics. This arises from the need to introduce a specific restriction to the formalism, either by symmetrizing the state or, as an alternative, by antisymmetrizing it. In other words, it seems that to address the notion of statistical identity, it is necessary to incorporate *ad hoc* a postulate that complements the formalism and aligns it with the statistically observed behavior. Clearly, with this interpretation, it becomes challenging to accept any kind of trans-statistics in a realistic sense, whether in terms of identity or behavior. However, as will be shown later, the argument developed in this article seeks to challenge this interpretation, relativize the notion of statistical identity, and allow a realistic understanding of trans-statistical behavior.

3. THE TRANS-STATISTICAL BEHAVIOR

In this presentation, the term "trans-statistical behavior" refers to a particular phenomenon that has captured the attention of physicists for several decades. It pertains to the fact that, under certain circumstances, a fermion system unexpectedly exhibits bosonic statistical behavior. It is argued that pairs of fermions can behave as composite bosons, also called cobosons. It's important to note that we are not referring to the already known cases where a strong nuclear interaction between fermions is observed, such as in a set of protons and neutrons (half-integer spin particles) that interact through the strong nuclear force to form an atomic nucleus. For instance, two protons and two neutrons can make up a Helium-4 nucleus. These atomic nuclei are composite systems that can display empirically verifiable bosonic behavior, such as superfluidity (Brooks & Donnelly 1977).

On the other hand, trans-statistical behavior does not stem from the strong nuclear interaction between fermions, and therefore, each subsystem can be dynamically independent. This feature makes the phenomenon theoretically intriguing since, according to the symmetrization postulate, the state of a composite system must be symmetrized or antisymmetrized *ab initio*, while the dynamic law of quantum mechanics does not allow a symmetric state to evolve into an antisymmetric state and vice versa. The symmetry or antisymmetry of the state of an isolated composite system must be maintained in any unitary dynamic evolution. Law (2005) addressed this phenomenon and found that the degree of entanglement between fermions determines to what extent a fermion system acts as a coboson system, and that interactions are not essential. If they exist, they simply intensify quantum correlations, which do seem to be crucial for the emergence of bosonic behavior. Subsequently, Chudzicki *et al.* (2010) and Tichy *et al.* (2014) achieved a generalization of Law's approach using creation and annihilation operators. This phenomenon is also relevant due to its applications in areas such as quantum information processing (Gigena & Rossignoli 2015), Bose-Einstein condensates (Avancini *et al*. 2003; Rombouts *et al.* 2003), excitons (Combescot & Tanguy 2001), Cooper pairs in superconductors (Belkhir & Randeria 1992), and confined Wigner molecules (Cuestas *et al*. 2020).

Generally, in the field of physics, there is a tendency to accept that trans-statistical behavior cannot be interpreted in a realistic manner. The prevailing interpretation holds that attributing bosonic behavior to a fermion system is merely an empirically adequate description, based on the use of idealized models and, therefore, inherently approximate (see, for example, Tichy *et al.* 2014). To illustrate this point, consider the perpetual motion inferred from models used to analyze pendulums. Physicists understand that perpetual motion is not expected in a real pendulum, which only behaves approximately in relation to the ideal model. Similarly, the bosonic behavior of a fermion system belongs, in a strict sense, to an ideal model and applies to real systems only approximately. Designating the behavior of a fermion system as bosonic is simply a useful description. This interpretation is supported by previously mentioned studies that link trans-statistical behavior to the degree of entanglement between the fermions of each pair. Thus, bosonic behavior, in a strict sense, would be viable only in an ideal situation where the degree of entanglement is total, that is, when $K = M$, where K is the Schmidt number and M is the number of modes that contain the Schmidt decomposition of the state (see Law 2005). It is considered, therefore, that the statistical behavior of a fermion system resembles bosonic behavior, but without being bosonic in a strict sense. The transition from one statistical behavior to another is understood, then, as a mere useful description without an *in re* reference. Consequently, the ontological status of cobosons is diminished. They are considered as virtual particles or quasiparticles, not comparable to conventional bosons, which are assumed to be elementary.

4. TPS APPROACH

A Tensor Product Structure (TPS) is a specific way (among several possible) to factorize or divide the Hilbert space, which represents a system, into subspaces that represent its subsystems. For example, consider a system U with an associated Hamiltonian H_U and eigenstates $|N\rangle$ such that $H_U|N\rangle = E_N|N\rangle$, where E_N are the possible values of the energy observable. The eigenstates $|N\rangle$ constitute a basis that generates the Hilbert space $\mathcal{H}_U$, representing the system U, in which the possible states $|\psi\rangle$ of U can be defined. Suppose it is possible to factorize the eigenstates $|N\rangle$ through the tensor product

$|m_i\rangle \otimes |m_{ii}\rangle = |N\rangle$. This way of decomposing $|N\rangle$ suggests that the system U is composed of two subsystems with states $|m_i\rangle$ and $|m_{ii}\rangle$. Under certain conditions, it is possible to define an H_i operator that acts as the Hamiltonian of the first particle so that the vectors $|m_i\rangle$ are eigenstates of this operator, such that $H_i|m_i\rangle = e_{m_i}|m_i\rangle$, while the vectors $|m_{ii}\rangle$ are the operator's eigenstates H_{ii} such that $H_{ii}|m_{ii}\rangle = e_{m_{ii}}|m_{ii}\rangle$, with $H_i \otimes I_{ii} + I_i \otimes H_{ii} = H_U$ (for the sake of simplicity, it is assumed that there is no interaction in this decomposition). Consequently, the energy values e_{m_i} and $e_{m_{ii}}$ are such that $e_{m_i} + e_{m_{ii}} = E_N$. The eigenstates $|m_i\rangle$ and $|m_{ii}\rangle$ respectively generate the subspaces $\mathcal{H}_i$ and $\mathcal{H}_{ii}$ so that $\mathcal{H}_i \otimes \mathcal{H}_{ii} = \mathcal{H}_U$. The subspace $\mathcal{H}_i$ represents the subsystem S_i and the subspace $\mathcal{H}_{ii}$ represents the subsystem S_{ii} so that $S_i \cup S_{ii} = U$. In this way, a tensor product structure represents one way, among many mathematically possible, to decompose a system into subsystems.

In a manner analogous, if the focus is shifted from the state to the observables, as is done in the algebraic formalism, it's possible to decompose a system into subsystems by breaking down the algebra of observables that represents it into subalgebras. Consider a system U with observables O_U that form an algebra $\mathcal{O}_U$ such that $O_U \in \mathcal{O}_U$. In this formalism, the state of the system is represented by a functional ρ acting on the operators in such a way that $\rho = |\psi\rangle\langle\psi|$. Now, a partition $\mathcal{O}_U = \mathcal{O}_i \vee \mathcal{O}_{ii}$ of the algebra $\mathcal{O}_U$ is proposed, such that for each $O_U \in \mathcal{O}_U$ it holds that $O_U = O_i \otimes I_{ii} + I_i \otimes O_{ii}$ with I_i and I_{ii} as identity operators. The observables O_i form a subalgebra $\mathcal{O}_i$ included in $\mathcal{O}_U$ ($\mathcal{O}_i \subset \mathcal{O}_U$) such that $O_i \in \mathcal{O}_i$, while observables O_{ii} form the subalgebra $\mathcal{O}_{ii}$ included in $\mathcal{O}_U$ ($\mathcal{O}_{ii} \subset \mathcal{O}_U$) such that $O_{ii} \in \mathcal{O}_{ii}$. Ultimately, the subalgebra $\mathcal{O}_i$

defines the subsystem S_i and $\mathcal{O}_{ii}$ defines the subsystem S_{ii} such that $S_i \cup S_{ii} = U$.

In this article, "TPS approach" refers to a line of research developed by various authors, who have examined the relativity of certain notions closely linked to quantum formalism in relation to the prior specification of a tensor product structure for a system. Thus, concepts such as the entanglement of quantum states or the separability between subsystems have been reconsidered from this perspective with notable results. Harshman and Wickramasekara (2007) emphasized the diversity of TPSs that a system can have, highlighting those that allow each subsystem to undergo both global symmetry transformations and dynamic transformations. These are termed by the authors as "symmetry invariant" and "dynamically invariant" TPSs. These TPSs are of special interest because the subsystems defined by them respect the symmetries of the Galilean group and have a unitary dynamic evolution.

Earman (2015), employing the algebraic formalism, placed emphasis on the relativity and even the ambiguity of the concept of *entanglement*. He posited that the entanglement of a system's state, delineated by its observable algebra, is inherently an entanglement relative to a decomposition of the algebra into subalgebras. A quantum state might be entangled with respect to one specific decomposition, yet be factorizable in relation to others. Absent a criterion that determines which decompositions ought to be favored, the concept of entanglement, in Earman's view, remains equivocal.

Zanardi (2001) and Dugić and Jeknić (2008) focused on the relativity of the notion of separability between subsystems. Zanardi attempted to circumvent the ambiguity of the separability notion by selecting those subalgebras of operators that represent a set of operationally accessible observables. These represent "real" subsystems as opposed to "virtual" subsystems, whose observables could not be measured. Dugić and Jeknić sought criteria to distinguish between "real" and "virtual" subsystems from the perspective of quantum decoherence and quantum information. However, they acknowledge that not only the notion of separability between subsystems, but also the very notion of a system should be relativized.

The intriguing question about the nature of systems (*what is a system?* 2006), posed by Dugić and Jeknić, might arise when taking the TPS approach beyond the purely physical realm, considering its ontological implications. This article proposes to address the question of the nature of systems as follows: ae physical systems individual entities or, at least, objects with clear identity conditions, such that the choice of TPSs is not only mathematically viable but also physically relevant? Or is it possible that physical systems lack, even at an ontological level, identity conditions that allow for finding physical criteria to select the appropriate partition? Later on, this article will advocate for the idea of an equivalence between the different ways of partitioning the Hilbert space, based on an ontology of possible properties for quantum systems. Through a model of trans-statistical behavior introduced below, the equivalence between a partition that "separates" ordinary fermions and another that "separates" composite bosons or cobosons will be examined.

5. A Toy Model

In this section, we propose a toy model for four fermions and two cobosons, which can be easily generalized to any even number of fermions. We will decompose a system using two distinct TPSs. Through the first one, subsystems with fermionic identity will be derived, and through the second, subsystems with bosonic identity. It is important to note that, unlike the models used in previously mentioned physical studies, our model does not rely on an experimental context to assess the conditions of trans-statistics. Instead, it explores an inherent possibility in the mathematical formulation of the theory. As will be shown, one of the strengths of the model might be its ability to determine the fermionic or bosonic identity of a total system without the need to alter its state according to the demands of the symmetrization postulate (SP). We suggest that this model indicates a new finding within the TPS approach: not only are the notions of entanglement and separability relative to the prior specification of the partition, but also the notion of statistical identity of a composite system becomes relative to the TPS from this perspective.

5.1 TENSOR PRODUCT STRUCTURE ALPHA (TPS OR PARTITION A)

Consider a system U associated with a Hamiltonian H with eigenstates $|N\rangle$ that generate a Hilbert space $\mathcal{H}$. The possible states $|\psi\rangle \in \mathcal{H}$ of the system U can generally be written as:

$$|\psi\rangle = \sum_N c_N |N\rangle$$ MERGEFORMAT

In this model, the system is composed of four non-interacting spin-½ particles, thus there is an automatically defined partition which will be called TPS A (alpha). To make this explicit, the eigenstates are factorized $|N\rangle$ by means of the tensor product $|n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \otimes |n_4\rangle = N$. Where $|n_1\rangle$, $|n_2\rangle$, $|n_3\rangle$ and $|n_4\rangle$ are the eigenstates of the Hamiltonians H_1, H_2, H_3 and H_4 for each particle respectively and generate the subspaces $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$ such that $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 = \mathcal{H}_U$. The subspace $\mathcal{H}_1$ represents the subsystem S_1, the subspace $\mathcal{H}_2$ represents the subsystem S_2, the subspace $\mathcal{H}_3$ represents the subsystem S_3 and the subspace $\mathcal{H}_4$ represents the subsystem S_4, such that $S_1 \cup S_2 \cup S_3 \cup S_4 = U$. The possible states $|\psi_1\rangle \in \mathcal{H}_1$ of the subsystem S_1 are written as $|\psi_1\rangle = \sum_{n_1} c_{n_1} |n_1\rangle$; the possible states $|\psi_2\rangle \in \mathcal{H}_2$ of the subsystem S_2 are written as $|\psi_2\rangle = \sum_{n_2} c_{n_2} |n_2\rangle$; the possible states

$|\psi_3\rangle \in \mathcal{H}_3$ of the subsystem S_3 are written as $|\psi_3\rangle = \sum_{n_3} c_{n_3} |n_3\rangle$ and the possible states $|\psi_4\rangle \in \mathcal{H}_4$ of the subsystem S_4 are written as $|\psi_4\rangle = \sum_{n_4} c_{n_4} |n_4\rangle$. The possible states $|\psi\rangle \in \mathcal{H}$ of the subsystem U can now be rewritten in the following way:

$$|\psi\rangle = \sum_{n_1,n_2,n_3,n_4} c_{n_1,n_2,n_3,n_4} |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \otimes |n_4\rangle$$ MERGEFORMAT

Now, a first series of permutation operators that exchange the states of only two of the subsystems is defined:

$$\begin{aligned}
P_{2134}|N\rangle &= |n_2\rangle \otimes |n_1\rangle \otimes |n_3\rangle \otimes |n_4\rangle \\
P_{3214}|N\rangle &= |n_3\rangle \otimes |n_2\rangle \otimes |n_1\rangle \otimes |n_4\rangle \\
P_{4231}|N\rangle &= |n_4\rangle \otimes |n_2\rangle \otimes |n_3\rangle \otimes |n_1\rangle \\
P_{1324}|N\rangle &= |n_1\rangle \otimes |n_3\rangle \otimes |n_2\rangle \otimes |n_4\rangle \\
P_{1432}|N\rangle &= |n_1\rangle \otimes |n_4\rangle \otimes |n_3\rangle \otimes |n_2\rangle \\
P_{1243}|N\rangle &= |n_1\rangle \otimes |n_2\rangle \otimes |n_4\rangle \otimes |n_3\rangle
\end{aligned}$$ MERGEFORMAT

Since the subsystems S_1, S_2, S_3 and S_4 are fermions, the state $|\psi\rangle$ of the total system U must be antisymmetric with respect to a permutation. That is:

$$
\begin{aligned}
P_{2134}|\psi\rangle &= -|\psi\rangle \\
P_{3214}|\psi\rangle &= -|\psi\rangle \\
P_{4231}|\psi\rangle &= -|\psi\rangle \\
P_{1324}|\psi\rangle &= -|\psi\rangle \\
P_{1432}|\psi\rangle &= -|\psi\rangle \\
P_{1243}|\psi\rangle &= -|\psi\rangle
\end{aligned}
$$

MERGEFORMAT

To ensure that the state $|\psi\rangle$ is fully antisymmetric, the state $|\psi\rangle$ must be antisymmetrized using an antisymmetrization operator

$$A = \frac{1}{N!}\sum_{\alpha} \pm P_{\alpha} \tag{9}$$

that is constructed from all the permutation operators P_{α}, those defined in (7) along with other second and third-order permutators (see details in Ballentine 1998). In this way, A allows for the antisymmetrization of $|\psi\rangle$ to obtain the antisymmetric state $|\psi_A\rangle$ that ensures the fermionic identity of the subsystems S_1, S_2, S_3 and S_4 of the composite system U. Thus,

$$A|\psi\rangle = |\psi_A\rangle \tag{10}$$

and in this way satisfy the indistinguishability postulate PIst (eq. 1), as expected from a system of indistinguishable particles.

PI$_{\text{st}}$: $\langle\psi'|O|\psi'\rangle = \langle\psi|O|\psi\rangle$ siendo $|\psi'\rangle = P|\psi\rangle$

$$\langle\psi_A|P_{\alpha}^{\dagger}OP_{\alpha}|\psi_A\rangle = (\pm 1)^2\langle\psi_A|O|\psi_A\rangle = \langle\psi_A|O|\psi_A\rangle$$

Where P_α represents any permutation operator. As can be seen, the squared eigenvalues (1) or (-1) have a neutral effect.

5.2 Tensor Product Structure Beta (TPS or partition B)

Consider the same system U associated with the same Hamiltonian H with the same eigenstates $|N\rangle$ that generate a Hilbert space $\mathcal{H}$. Now define a new partition TPS B (beta) by factorizing the eigenstates $|N\rangle$ in a different way than was done in TPS A, now using the tensor product $|m_i\rangle \otimes |m_{ii}\rangle = N$ such that $|m_i\rangle = |n_1\rangle \otimes |n_2\rangle$ and $|m_{ii}\rangle = |n_3\rangle \otimes |n_4\rangle$. The bases $|m_i\rangle$ and $|m_{ii}\rangle$ respectively generate the subspaces $\mathcal{H}_i$ and $\mathcal{H}_{ii}$ such that $\mathcal{H}_i \otimes \mathcal{H}_{ii} = \mathcal{H}_U$. The subspace $\mathcal{H}_i$ represents the subsystem S_i and the subspace $\mathcal{H}_{ii}$ the subsystem, such that $S_i \cup S_{ii} = U$. Given that

- $|m_i\rangle = |n_1\rangle \otimes |n_2\rangle$, it follows that $\mathcal{H}_i = \mathcal{H}_1 \otimes \mathcal{H}_2$, hence $S_i = S_1 \cup S_2$.
- $|m_{ii}\rangle = |n_3\rangle \otimes |n_4\rangle$, it follows that $\mathcal{H}_{ii} = \mathcal{H}_3 \otimes \mathcal{H}_4$, hence $S_{ii} = S_3 \cup S_4$.

This establishes a clear correspondence between both partitions. The possible states $|\psi_i\rangle \in \mathcal{H}_i$ of subsystem S_i are written as $|\psi_i\rangle = \sum_{m_i} c_{m_i} |m_i\rangle$ and the possible states $|\psi_{ii}\rangle \in \mathcal{H}_{ii}$ of subsystem S_{ii} are written $|\psi_{ii}\rangle = \sum_{m_{ii}} c_{m_{ii}} |m_{ii}\rangle$. The possible states $|\psi\rangle \in \mathcal{H}$ of the system U can now be rewritten in the following way:

$$|\psi\rangle = \sum_{m_i, m_{ii}} c_{m_i, m_{ii}} |m_i\rangle \otimes |m_{ii}\rangle \tag{12}$$

In this case, it is possible to define a single permutation operator corresponding to TPS B:

$$P_{ii-i}|N\rangle = |m_{ii}\rangle \otimes |m_i\rangle \tag{13}$$

Since there is a correspondence between the states in TPS A and states in TPS B such that $|m_i\rangle = |n_1\rangle \otimes |n_2\rangle$ and $|m_{ii}\rangle = |n_3\rangle \otimes |n_4\rangle$ exist, there is also a correspondence between the operator P_{ii-i} and one of the permutation operators of TPS A. Namely,

$$P_{ii-i}|N\rangle = |m_{ii}\rangle \otimes |m_i\rangle = |n_3\rangle \otimes |n_4\rangle \otimes |n_1\rangle \otimes |n_2\rangle = P_{3214}P_{1432}|N\rangle \tag{14}$$

Thus,

$$P_{ii-i} = P_{3214}P_{1432} \tag{15}$$

When considering partition A, it was assumed that the subsystems S_1, S_2, S_3 and S_4 are fermions, and therefore the state of the system U was antisymmetrized to obtain $|\psi_A\rangle$. Without abandoning this assumption in the following, let's see how the operator P_{ii-i} acts on the antisymmetric state $|\psi_A\rangle$.

$$P_{ii-i}|\psi_A\rangle = P_{3214}P_{1432}|\psi_A\rangle = |\psi_A\rangle \tag{16}$$

This result is significant because the same state $|\psi_A\rangle$ turns out to be symmetric with respect to the operator P_{ii-i}. Thus, the notation that introduces the subscript A when the state is antisymmetric ($|\psi_A\rangle$) and the subscript S when it is symmetric ($|\psi_S\rangle$) becomes insufficient because this example shows that the same state can be symmetric or antisymmetric depending on which TPS is considered. Indeed, in our example, the state of the system is antisymmetric if analyzed from TPS A but is symmetric if analyzed from TPS B, that is:

$$|\psi_{A(\mathrm{TPSA})}\rangle = |\psi_{S(\mathrm{TPSA})}\rangle \tag{17}$$

This means that precisely because it was assumed that the subsystems S_1, S_2, S_3 and S_4 of partition A are fermions, it must be admitted that the subsystems S_i and S_{ii} of partition B can be identified as bosons.

5.3 The Relativity of the Notion of Statistical Identity and the Ontological Status of Cobosons. Partial Conclusions.

The aforementioned result leads to the assertion that the system U with respect to TPS A is a fermionic system and, at the same time, that the system U with respect to TPS B is a bosonic system, without the need to modify its state *(i.e.* without reapplying SP to obtain the bosonic identity). The model proposes that the statistical identity of a composite system, which can be divided into indistinguishable subsystems, depends on the selected tensor product structure (or simply *TPS-relative*). This suggests that for any system composed of an even number of fermions, there exists an alternative TPS where each pair of fermions is considered as a single subsystem. This perspective could be added to the results previously derived from the TPS approach. Specifically, in addition to the relativity in the notions of separability and entanglement, we could now consider statistical identity as another relative property. If this interpretation is correct, the fermionic or bosonic identity of a system that can be divided into indistinguishable subsystems should be understood in terms relative to the TPS. In this framework, the designations "being a fermion" and "being a boson," applicable to indistinguishable subsystems, would be interpreted as relational properties. Thus, a subsystem would have a specific statistical identity in relation to other subsystems within the same partition, rather than possessing an identity in a more absolute sense. Therefore, it would not be essential to consider the properties of "being a fermion" or "being a boson" as fixed categories, and hence, it would not be necessary to adjust them to the formalism through a specific postulate, such as the *ad hoc* (SP).

The toy model suggests that the condition of possibility for trans-statistical behavior lies in this notion of TPS-relativized statistical identity. Thus, trans-statistical behavior would not depend solely on specific physical conditions modeled approximately (e.g., the degree of entanglement of the states of the fermions in each pair), but also on a form of trans-statistics present at a more fundamental level: the qualitative identity of the systems. Therefore, it could be interpreted that trans-statistical behavior is not merely a convenient description of a certain phenomenon. This behavior could be linked to the conditions under which the qualitative identity of indistinguishable quantum systems is established, and, from this perspective, could be interpreted in a realistic manner.

However, the partial conclusions that might be derived from the toy model based on the TPS approach should be considered with caution for the time being. Although the model succeeds in avoiding the repeated use of the symmetrization postulate (SP) to obtain a system of bosons from a system of fermions, it still relies on the formalism of Hilbert spaces. In this formalism, the state vector identifies a quantum system, and the operators representing observables act in an already defined state space. These particularities of the Hilbert space formalism align it with an ontology of individual objects that possess inherent properties. This ontology does not allow for the full relativization of the notion of separability, and therefore blocks the path towards a relativized notion of statistical identity. Among all the available TPSs, some must give rise to subsystems that coincide with objects that maintain conditions of identity. Hence, for this ontology, the TPSs that define more basic subsystems will have priority. And since in the toy model the state vectors that define the subsystems of TPS B can be expressed as tensor products of the state vectors of TPS A (and not vice versa), the natural interpretation is that the total system is fermionic and that TPS A has ontological primacy over TPS B.

Therefore, at this stage of the proposal, where a model still based on the formalism of Hilbert spaces has been presented, the mentioned hierarchy between TPSs compels us to grant ontological priority to the subsystems of TPS A, and to reconsider what was previously stated about the relational character of the notion of statistical identity. The subsystems are fermions that can act like bosons when considered in pairs; however, these pairs are not genuine bosons, but cobosons. In other words, the statistical identity of the total system (fermionic) is determined by identity conditions linked to the subsystems in their individuality, disregarding the relationships that exist between subsystems within a composite system. From an ontology of individuals, one must assert: "this is a system of fermions that, although it is not a system of bosons, can behave as such." Fermions are concrete systems, while cobosons are merely apparent particles, in line with what is commonly assumed in the physical context.

However, a more detailed response to the dilemma of the relative or absolute nature of statistical identity (and, in relation to this, the problem of the ontological status of cobosons and the realistic interpretation of trans-statistical behavior) will be provided after introducing the algebraic version of the toy model and its interpretation based on the ontology of properties, suggested by the algebraic formalism of quantum mechanics.

6. THE ONTOLOGY OF PROPERTIES

Traditional metaphysics developed the ontological notion of individual object, pertinent to classical physical systems. This notion of individual object is linked to the semantic notion of singular reference and the logical notion of the subject of predication (Laycock 2014). Therefore, the idea of individual object is inherently complemented by the ontological notion of properties. In other words, a set of properties belongs to individual object and is predicated of it. Such an object is distinguished by possessing conditions of identity that differentiate it synchronically from other objects and allow its diachronic reidentification despite changes in its properties over time. Some approaches maintain that individuality of an object is based on a principle that goes beyond properties, such as substance (transcendental individuality). Others argue that individuality of an object lies solely in its properties (bundle individuality). Those who see individual object as a bundle of properties take on the additional challenge of establishing criteria for synchronic and diachronic identity based purely on properties. For diachronic identity, the object's spatiotemporal trajectory is commonly used. As for synchronic identity, some variant of Leibniz's Principle of Identity of Indiscernibles (PII) is generally adopted. The PII states that two objects with indistinguishable properties are numerically identical, that is, they must be considered in the ontological realm as a single object. Different interpretations of the PII derive from considering different sets of properties (monadic, relational, etc.) as essential for the criterion.

In section 2, the indistinguishability of quantum systems was alluded to. The statistics governing quantum systems of the same type led us to recognize that if these systems are objects in any sense, they are so in an atypical way, given that they lack conditions of identity that categorize them as individuals. Specifically, they do not obey the PII, since in quantum mechanics, systems with indistinguishable properties present only numerical difference. Moreover, the so-called quantum contextuality, which prevents all observables of a system from having simultaneously defined values, challenges the traditional principle of omnimodal determination expected of any individual object, and hinders the possibility of using the spatiotemporal trajectory as a criterion for diachronic identity. Finally, there are situations in which quantum systems violate the principle of locality, which is expected to be fulfilled by every individual object. These three ontological challenges posed by quantum systems led, even in the initial phases of the formulation of quantum mechanics (see Weyl 1931), to the indication that quantum systems do not possess the typical conditions of identity of individual objects. This perception was consolidated in what is called the "received view" on the ontological status of quantum systems, which subsequently led to the development of alternative formal systems to represent non-individual objects (such as Krause's quasi-set theory of 1992). Recently, various authors have questioned the received view, arguing that the notion of an individual object can be reclaimed if the PII is discarded, or at least some of its more restrictive versions. Van Fraassen (1985) suggests abandoning the principle of equiprobability; French (1989) postulates that neither symmetric nor antisymmetric states are ontologically viable but physically unreachable; Muller and Saunders (2008) explore ways to identify relational properties that differentiate between fermions. In this context, some defenders of modal interpretations of quantum mechanics have outlined an innovative ontology of possible properties, in which quantum systems lack conditions of identity that define a subject of predication (see Lombardi & Castagnino 2008, da Costa, Lombardi & Lastiri 2013; da Costa & Lombardi 2014; Lombardi & Dieks 2016). This proposal aligns with the revolutionary character of the received view. According to this ontology, quantum systems cannot be seen as individuals, and not even as objects (see Lombardi & Pasqualini 2022). It will be discussed how, in this proposal, the notion of physical separability between systems is not ontologically predefined, but emerges from pragmatic stipulations in the realm of physics. Originally conceived to address the traditional ontological challenges of indistinguishability, contextuality, and non-locality, it is argued that this ontology is also the most appropriate for interpreting (in a realistic approach) the trans-statistical behavior that is the focus of this article.

6.1 The Ontology of Properties and Algebraic Formalism

Standard presentations of quantum mechanics utilize the Hilbert space formalism for the mathematical representation of physical systems. A Hilbert space is structured from a set of complex vectors, where each vector symbolizes a possible state of the system. Physical observables are represented by operators acting on these pre-established state vectors. This logical primacy of the state space over the operators symbolizing observables points to an ontology of individuals, in which systems are defined by their state space and identified by their state vector. Subsequently, they acquire the properties associated with the operators (cf. Ballentine 1998 234-235).

However, it is also possible to employ the algebraic formalism in quantum mechanics, which is mathematically equivalent to the former. In this formalism, quantum systems are immediately represented by an operator algebra that represents their observables. Quantum states are represented by functionals acting on the previously defined operators. In this case, the logical priority of the operators representing the observables over the functional representing the state suggests an ontology of properties in which quantum systems are immediately defined by their properties, corresponding to the operators, without any substrate. The state of the system lacks an ontological correlate, being merely a mathematical device that encodes the probability distributions among possible values corresponding to each observable. (Ballentine 1998 48).

To be more specific, the ontology of possible properties is defined through the following semantic correspondences (cf. Lombardi & Pasqualini 2022):

- The algebra of *self-adjoint operators* represents the set of *physical observables* that define a quantum system, which in turn corresponds to the set of *instances of universal type- properties* in the ontological domain.

- The *eigenvalues* of the self-adjoint operators represent *possible physical values*, which in turn correspond to the set of *possible case-properties* belonging to each instance of type-property.
- *Probability functions* represent *probability distributions* for each physical observable, which in turn correspond to the *ontological propensities* for the actualization of each possible case-property.
- *Functionals* over the algebra of observables represent *physical states*. They are simple devices that assign a probability distribution to each observable of a singular system and, therefore, from the ontological point of view, encode the ontological propensities for all possible case-properties of the system.

It should be noted that the term "possible properties" is used because, as mentioned, this ontology was conceived in the context of modal interpretations, where the category of possibility has ontological content. However, nothing prevents its application outside of modal interpretations. In this ontology, quantum systems are considered bundles of possible properties. This proposal aligns with the traditional approach to bundle individuality, but with the significant distinction that, unlike the traditional theory of bundles of actual properties, it does not seek to impose the conditions of identity characteristic of individual objects onto properties. In addition to the challenges presented by the ontological treatment of indistinguishable systems, a quantum system could not be considered a bundle of actual properties due to the limitations associated with quantum contextuality, clearly established in the Kochen and Specker theorem (1967). On the contrary, this ontology intentionally seeks to dismantle any ontologically grounded conditions of identity. Regarding possible properties, the PII is not incorrect; it simply does not apply, as there is no impediment to the numerical difference between two formal objects with identical possible properties. To further differentiate from the traditional bundle theory, the term "cluster" of possible properties has been adopted to refer to quantum systems in the ontological domain (Lombardi & Pasqualini 2022).

As observed, the conventional image of quantum systems as particles with relatively stable conditions of identity is diametrically opposed to the image of quantum systems provided by this ontology. According to it, quantum subsystems, considered individually, do not possess conditions of identity that are maintained when integrated into compounds. Clusters of properties can be aggregated to form new clusters, without the originals being reidentifiable. In turn, a cluster can be broken down in various ways, without any having ontological preeminence. Specifically, revisiting the TPS approach mentioned previously, if a system can be mathematically divided in various ways, the ontology does not favor any division as representative of conditions of identity that establish outstanding singular references. The distinction between "real" and "virtual" systems is blurred. This ontology, therefore, provides a clear ontological sense to a notion of separability that is completely relativized, as suggested by the TPS approach. From the postulates of this ontology, neither the traditional atomistic perspective, where all physical reality is constructed bottom-up from elementary systems with absolute conditions of identity, nor a holistic, top-down view, where the identity of the parts is completely relativized to the properties of the whole, is derived. This does not prevent that, in physical practice and in the interpretations of quantum mechanics, it is convenient to use certain partitions rather than others, establishing relatively stable conditions of identity for certain systems. However, from this ontology, such preference for certain divisions has no ontological foundation but results from stipulations oriented towards practical or interpretative goals.

ONTOLOGY OF PROPERTIES AND INDISTINGUISHABILITY

From an ontology of properties, where quantum systems are defined not by their state space or state vector, but by their possible type-properties and case-properties, it is necessary to reformulate the standard postulate of indistinguishability. In this context, it is the quantum observables that must be invariant under the available permutation operators. This reformulation is known as the principle of indistinguishability over observables IP_{obs}:

IP_{obs}: If the operators O represent the observables of the composite system whose subsystems are indistinguishable, then the expectation value of any observable represented by an operator O must be the same for O and for any permutation O'.

IP_{obs}: $\langle\psi|O'|\psi\rangle = \langle\psi|O|\psi\rangle$ where $O' = P^{\dagger}OP$

To satisfy IP_{obs}, there is no longer a need to symmetrize or antisymmetrize the state of the system but to directly symmetrize the system's observables. The reformulated SP reads:

SP_{obs}: A system of multiple identical particles must be represented by an algebra of symmetric operators $O_{sim} \in \mathcal{O}_{sim}$, where symmetry is defined in terms of permutation operators P.

$$O' = P^{\dagger}OP = (\pm 1)^2 O = O_{sim} \qquad (19)$$

This condition imposed on the observables includes both bosons and fermions. It is easy to recognize this if we refer to the restriction that is usually imposed on the states. In the case of bosons, it is required that the wave function be symmetric, that is:

$$|\psi_S\rangle = P|\psi_S\rangle \qquad (20)$$

Such that,

$$\langle\psi_S|O_{sim}|\psi_S\rangle = \langle\psi_S|P^{\dagger}OP|\psi_S\rangle = \langle\psi_S|O|\psi_S\rangle$$

And in the case of fermions, it is required that $|\psi_A\rangle = -P|\psi_A\rangle$, such that

$$\langle\psi_A|O_{sim}|\psi_A\rangle=\langle\psi_A|P^\dagger OP|\psi_A\rangle=\left(-\langle\psi_A|\right)O\left(-|\psi_A\rangle\right)=\langle\psi_A|O|\psi_A\rangle$$

Thus, the condition expressed in (19) that defines the symmetric observables O_{sim} includes both cases. To obtain observables with a defined statistical identity, symmetrization operators S (for the case of bosons) or antisymmetrization A (for the case of fermions) should be applied to the observables

$$\begin{aligned} S^\dagger OS &= O_B \\ A^\dagger OA &= O_F \end{aligned} \tag{22}$$

The symmetric operators O_B form the bosonic algebra $\mathcal{O}_B$, while the symmetric operators O_F form the fermionic algebra $\mathcal{O}_F$. It should be clarified that the operators O_F, although they have been the result of applying an antisymmetrization operator, turn out to be symmetric and not antisymmetric since, when permutation operators are applied to them, the possible eigenvalue (-1) appears squared. This is:

$$P^\dagger O_F P=(\pm 1)^2\, O_F=O_F \tag{23}$$

Thus, the expected mean values for bosonic or fermionic systems are obtained without the need to symmetrize or antisymmetrize the state.

$$\begin{aligned} \langle O\rangle_{|\psi_S\rangle} &= \langle\psi_S|O|\psi_S\rangle=\langle\psi|S^\dagger OS|\psi\rangle=\langle\psi|O_B|\psi\rangle=\langle O_B\rangle_{|\psi\rangle}=\mathrm{Tr}\left(\rho O_B\right) \\ \langle O\rangle_{|\psi_A\rangle} &= \langle\psi_A|O|\psi_A\rangle=\langle\psi|A^\dagger OA|\psi\rangle=\langle\psi|O_F|\psi\rangle=\langle O_F\rangle_{|\psi\rangle}=\mathrm{Tr}\left(\rho O_F\right) \end{aligned} \tag{24}$$

From this ontological perspective, the symmetrization postulate ceases to be an ad hoc addition and is ontologically grounded. When two or more clusters of identical possible properties are combined to form a single cluster, it is natural to expect that the resulting cluster will be symmetric. That is, the symmetric observables of the composite system do not distinguish between one subsystem and another. For example, consider two clusters h^1 and h^2 defined by different instances of identical observable algebras $\mathcal{O}_1 = \mathcal{O}_2$ such that $h^1 \square h^2$, where $\square$ is the indistinguishability relation. The indices 1 and 2 here do not imply individuality and could be arbitrarily interchanged. These two clusters are combined into a new composite cluster h^U such that $h^U = h^1 * h^2$, where $*$ is the aggregation operation. The algebra $\mathcal{O}_U = \mathcal{O}_1 \vee \mathcal{O}_2 = \mathcal{O}_2 \vee \mathcal{O}_1$ will define the cluster h^U. Then, the restriction on the observables $O_U \in \mathcal{O}_U$ required by SP$_{\text{obs}}$ must be carried out, so that the observables $O_U = \sum_{ij} k_{ij} \left(O_{1i} \otimes O_{2j} \right)$ are such that $O_{1i} \otimes O_{2j} = O_{2i} \otimes O_{1j}$, that is, they are invariant under the permutation of the clusters h^1 and h^2 (Fortin & Lombardi 2021).

7. Algebraic Version of the Model

Introduced the algebraic formalism along with the possible property ontology, the algebraic version of the toy model presented in section 5 is presented below. Consider a system U with observables O_U belonging to the algebra $\mathcal{O}_U$. The state of the system U will be the generic $\rho = |\psi\rangle\langle\psi|$ (without symmetrizing or antisymmetrizing).

The partition A (TPS A) $\mathcal{O}_U = \mathcal{O}_1 \vee \mathcal{O}_2 \vee \mathcal{O}_3 \vee \mathcal{O}_4$ of the algebra $\mathcal{O}_U$ is introduced such that for each $O_U \in \mathcal{O}_U$ it is the case that:

$$O_U = \sum_{ijkl} k_{ijkl} \left(O_{1i} \otimes O_{2j} \otimes O_{3k} \otimes O_{4l} \right) \qquad (25)$$

The observables O_1 belong to the subalgebra $\mathcal{O}_1 \subset \mathcal{O}_U$; the observables O_2 belong to the subalgebra $\mathcal{O}_2 \subset \mathcal{O}_U$; the observables O_3 belong to the subalgebra $\mathcal{O}_3 \subset \mathcal{O}_U$ and the observables O_4 belong to the subalgebra $\mathcal{O}_4 \subset \mathcal{O}_U$. Finally, the subalgebras $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ define respectively the subsystems S_1, S_2, S_3 and S_4 such that $S_1 \cup S_2 \cup S_3 \cup S_4 = U$.

Consequently, the first-order permutation operators for this partition can be defined as follows:

$$P_{2134}^{\dagger} O_U P_{2134} = \sum_{ijkl} k_{ijkl} \left(O_{2i} \otimes O_{1j} \otimes O_{3k} \otimes O_{4l} \right)$$
$$P_{3214}^{\dagger} O_U P_{3214} = \sum_{ijkl} k_{ijkl} \left(O_{3i} \otimes O_{2j} \otimes O_{1k} \otimes O_{4l} \right)$$
$$P_{4231}^{\dagger} O_U P_{4231} = \sum_{ijkl} k_{ijkl} \left(O_{4i} \otimes O_{2j} \otimes O_{3k} \otimes O_{1l} \right)$$
$$P_{1324}^{\dagger} O_U P_{1324} = \sum_{ijkl} k_{ijkl} \left(O_{1i} \otimes O_{3j} \otimes O_{2k} \otimes O_{4l} \right)$$
$$P_{1432}^{\dagger} O_U P_{1432} = \sum_{ijkl} k_{ijkl} \left(O_{1i} \otimes O_{4j} \otimes O_{3k} \otimes O_{2l} \right)$$
$$P_{1243}^{\dagger} O_U P_{1243} = \sum_{ijkl} k_{ijkl} \left(O_{1i} \otimes O_{2j} \otimes O_{4k} \otimes O_{3l} \right)$$

Partition B (beta) is introduced $\mathcal{O}_U = \mathcal{O}_{\mathrm{i}} \vee \mathcal{O}_{\mathrm{ii}}$ for the algebra $\mathcal{O}_U$ such that for each $O_U \in \mathcal{O}_U$ it holds that $O_U = \sum_{mn} k_{mn} \left(O_{\mathrm{i}m} \otimes O_{\mathrm{ii}n} \right)$. The observables O_{i} belong to the subalgebra $\mathcal{O}_{\mathrm{i}}$, while the observables O_{ii} belong to the subalgebra $\mathcal{O}_{\mathrm{ii}}$. The subalgebras $\mathcal{O}_{\mathrm{i}}$ and $\mathcal{O}_{\mathrm{ii}}$ define the subsystems S_i and S_{ii}

respectively, so that $S_i \cup S_{ii} = U$. Finally, the algebra $\mathcal{O}_{\mathrm{i}}$ has as subalgebras the already known $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $S_i = S_1 \cup S_2$, *i. e.* $\mathcal{O}_{\mathrm{i}} = \mathcal{O}_1 \vee \mathcal{O}_2$; and $\mathcal{O}_{\mathrm{ii}}$ has as subalgebras $\mathcal{O}_3$ and $\mathcal{O}_4$ such that $S_{ii} = S_3 \cup S_4$; *i. e.* $\mathcal{O}_{\mathrm{ii}} = \mathcal{O}_3 \vee \mathcal{O}_4$.

The only permutation operator corresponding to this partition is:

$$P_{ii-i}^{\dagger} O_U P_{ii-i} = \sum_{mn} k_{mn} \left(O_{\mathrm{ii}m} \otimes O_{\mathrm{i}n} \right) \qquad (27)$$

Since $\mathcal{O}_{\mathrm{i}} = \mathcal{O}_1 \vee \mathcal{O}_2$ and $\mathcal{O}_{\mathrm{ii}} = \mathcal{O}_3 \vee \mathcal{O}_4$, there is a correspondence between P_{ii-i} and one of the permutation operators of partition A. Namely

$$\sum_{mn} k_{mn} \left(O_{\mathrm{ii}m} \otimes O_{\mathrm{i}n} \right)_i = \sum_{ijkl} k_{ijkl} \left(O_{3i} \otimes O_{4j} \otimes O_{1k} \otimes O_{2l} \right)$$

$$P_{ii-i}^{\dagger} O_U P_{ii-i} = P_{3214}^{\dagger} P_{1432}^{\dagger} O_U P_{1432} P_{3214}$$

Thus,

$$P_{ii-i} = P_{1432} P_{3214} \qquad (29)$$

If it is assumed that the subsystems S_1, S_2, S_3 and S_4 are indistinguishable from each other, it should be the case, in order to satisfy IP$_{\text{obs}}$

$$\forall O_U \in \mathcal{O}_U, O_U = P_{\alpha}^{\dagger} O_U P_{\alpha} \qquad (30)$$

If additionally it is assumed that S_1, S_2, S_3 and S_4 are fermions, and in order to ensure that the observables representing the system U comply with the condition established in Eq. 30, the already defined antisymmetrization operator A must be applied to the observables O_U

$$\forall O_F \in \mathcal{O}_F,\ O_F = A^{\dagger} O_U A \qquad (31)$$

Thus, the algebra $\mathcal{O}_F$ is obtained, which represents the system U composed of four fermions S_1, S_2, S_3 and S_4. The operation is analogous to applying the antisymmetrization operator A to the state. The IP$_{obs}$ is satisfied by this algebra since

IP$_{obs}$: $\langle \psi | O' | \psi \rangle = \langle \psi | O | \psi \rangle$ where $O' = P^{\dagger} O P$

$$\langle \psi | P_{\alpha}^{\dagger} O_F P_{\alpha} | \psi \rangle = (\pm 1)^2 \langle \psi | O_F | \psi \rangle = \langle \psi | O_F | \psi \rangle$$

Assuming now that the subsystems S_i and S_{ii} are also indistinguishable from each other, the following should be satisfied to meet IP$_{obs}$

$$\forall O_U \in \mathcal{O}_U, O_U = P_{ii-i}^{\dagger} O_U P_{ii-i} \qquad (33)$$

If additionally it is assumed that S_i and S_{ii} are bosons, and in order to ensure that the observables representing the system U comply with the condition established in eq. 33, a symmetrization operator $\tilde{S}$ should be applied to the observables O_U. The symmetrization operator $\tilde{S}$ is defined as follows:

$$\tilde{S} = \frac{1}{2}\left(I + P_{ii-i}\right) \qquad (34)$$

One would obtain an algebra of observables $\tilde{\mathcal{O}}_B$ such that

$$\forall \tilde{O}_B \in \tilde{\mathcal{O}}_B,\ \tilde{O}_B = \tilde{S}^{\dagger} O_U \tilde{S} \qquad (35)$$

However, if this operator were used, an undesirable result would be obtained, which is to obtain an algebra of observables $\tilde{\mathcal{O}}_B$ that would define U as a system different from the one previously defined by the algebra $\mathcal{O}_F$. We would no longer be talking about two different partitions of the same composite system but about two different composite systems. It would be the equivalent in the Hilbert space formalism of symmetrizing the state with respect exclusively to the operator P_{ii-i}, but the symmetries and antisymmetries with respect to the operators P_α would be lost.

The way to find a suitable algebra of observables, which satisfies eq. 33 without failing to satisfy eq. 30, is to define an algebra $\mathcal{O}_B = \mathcal{O}_F$ by making use of the same antisymmetrization operator A. The sought-after algebra is

$$\forall O_B \in \mathcal{O}_B,\ O_B = A^\dagger O_U A \qquad (36)$$

This is perfectly possible, as it has been shown that the action of the operator A should be understood as an antisymmetrization from partition A but as a symmetrization from partition B. Thus, the algebra $\mathcal{O}_B$ is obtained, which represents the system U composed of two bosons S_i and S_{ii}. The $\mathrm{IP_{obs}}$ is satisfied by this algebra because

$$\mathrm{IP_{obs}}:\ \left\langle \psi \middle| O' \middle| \psi \right\rangle = \left\langle \psi \middle| O \middle| \psi \right\rangle \text{ where } O' = P^\dagger O P$$

$$\left\langle \psi \middle| P_{ii-i}^\dagger O_B P_{ii-i} \middle| \psi \right\rangle = \left(1\right)^2 \left\langle \psi \middle| O_B \middle| \psi \right\rangle = \left\langle \psi \middle| O_B \middle| \psi \right\rangle$$

As a first corollary, it is obtained that the same algebra $\mathcal{O}_B = \mathcal{O}_F$ obtained from the antisymmetrization operator A defines the system U as fermionic or bosonic not in absolute terms but with respect to a certain partition; that is, U is fermionic with respect to partition A and bosonic with respect to B. This invites a change in notation to highlight the relativity of statistical identity with respect to the partition. Instead of $\mathcal{O}_B$ it will be said $\mathcal{O}_{\mathrm{B}}^{B}$, instead of $\mathcal{O}_F$ it will be said $\mathcal{O}_{\mathrm{A}}^{F}$. A similar result had been obtained through the Hilbert space version of the model, in which for a total system represented by the same state vector $|\psi_A\rangle$ we obtained both fermionic and bosonic identities.

The second corollary is that the use of the antisymmetrization operator A to obtain the algebra $\mathcal{O}_B = \mathcal{O}_F$ does not necessarily have to lead to the conclusion that the system U is fundamentally fermionic and that partition A has ontological priority over B. Instead, the use of the antisymmetrization operator A can be interpreted as a certain restriction on the algebra of observables, among the many possible ones, that defines a certain subalgebra and thereby defines a certain set of subsystems. This restriction is the analogue in the space of observables that from another point of view could be introduced from a restriction of the states accessible to the system. Specifically, the restriction lies in limiting the possible states of the system to those states that, being symmetric in partition B, become antisymmetric when viewed from partition A.

This result emerges as a latent possibility only in the algebraic version. Namely, we have an algebra $\mathcal{O}_{sim}$ composed of operators $O_{sim} = S^{\dagger} O_U S$ that perfectly adapts to the computation of mean values for both the fermionic and bosonic partitions and that could not be considered absolutely fermionic (since the symmetrization operator S) nor absolutely bosonic since S is built upon the permutation operators of the fermionic partition.

8. CONCLUSIONS

The adoption of a property ontology suggested by the algebraic formalism of quantum mechanics allows us to lift the caution that had forced us to suspend the partial conclusions derived from the Hilbert space version of the toy model. The ontological equivalence of the plurality of TPSs from an ontology in which quantum (sub)systems cannot be considered objects with identity conditions that can be preserved when partitioned or when entering into composition, prevents the conventional interpretation that bases the statistical identity of a composite system on certain intrinsic properties of those subsystems defined by a privileged TPS, which delimits systems considered elementary. From this ontology, by enabling a notion of TPS-relativized separability, a notion of TPS-relativized statistical identity is also made possible.

Indistinguishable subsystems are no longer, in a strict sense, either fermions or bosons, but they are in a relative sense. That is, the statistical identity of the composite system no longer depends on the identity conditions of each numerically distinct subsystem considered individually, but is attributed to the composite system in relation to a specific TPS and to each subsystem in relation to the rest of the subsystems of the partition. Recall that adopting a property ontology implies assuming a top-down approach, where we start with the total system as a unique cluster of properties, attenuating the notion of elementality. The statistical identity of the total system will be determined by the mutual relational properties of the subsystems defined by each TPS.

From the property ontology, it can finally be said: "this is a system of fermions with respect to partition A and it is a system of bosons with respect to partition B." By relativizing the fermionic or bosonic identity, a realistic interpretation of trans-statistical behavior is facilitated. In an ontology of individuals, the bosonic behavior of a system of fermions could only be apparent. However, from a property ontology, the trans-statistical behavior of a system that is fermionic or bosonic in a TPS-relative manner no longer depends solely on specific physical circumstances and can have an ontological foundation, linked to the conditions under which the qualitative identity of the systems is established. Moreover, cobosons turn out to be as real as the fermions that constitute them, and their ontological status is equated with that of conventional bosons.

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